Pressures and Temperatures in the deep Earth

1. Introduction

The following notes are probably more general and detailed than needed but we shall refer back to them during other parts of the class. Pressure is relatively easy to compute but temperature is much more difficult. We have some idea how temperature varies in a vigorously convecting region.

2. Forces acting on a body

In the Earth, both body forces (gravity) and surface forces are important. We consider a body with volume $V$ and surface $S$ and introduce the body force density $b$ and the traction vector $t$ such that the total body force acting on the body is

$$\int_V \rho b \, dV \quad (1)$$

and the total surface force acting on the body is

$$\int_S t \, dS \quad (2)$$

Note that $b$ is reckoned per unit mass and $t$ is reckoned per unit area. $t$ is most conveniently specified by introducing the stress tensor. If $\hat{n}$ is the normal to a surface then the traction acting on the plane with that normal is defined by

$$t = \hat{n} \cdot T \quad (3)$$

This equation defines the Cauchy stress tensor, $T$, which is the linear vector function which associates with each unit normal $\hat{n}$ the traction vector $t$ acting at the point across the surface whose normal is $\hat{n}$.

The mean normal pressure is defined as

$$p = -\frac{1}{3} (T_{kk}) \quad (4)$$

Note that this is invariant to rotations of the coordinate system. The minus sign arises because of our sign convention. $T_{ij}$ describes the surface force acting in the $j$'th direction on the surface with normal in the $i$'th direction. Thus $T_{i1}$ acts in the 1 direction on a plane with normal in the 1 direction and is a tensile stress (fig 1). A positive pressure is usually taken to be a compressive stress and so we have the minus sign in equation 4.

3. Material derivatives

Consider a particle initially at a position $X$ in a reference configuration ($t = 0$) which is at position $x$ at time $t$ and define the displacement as

$$x = X + s \quad (5)$$

In continuum mechanics, there are several ways of describing a configuration of particles. A material (or Lagrangian) description defines motion as

$$x = x(X, t)$$
This description essentially labels particles and is often the most natural description because conservation laws in mechanics apply to particles rather than to some fixed region of space.

In the spatial (or Eulerian) description, \( x \) and \( t \) are taken to be the independent variables and we have

\[
X = X(x, t)
\]

This description gives the initial position of the particles now (at time \( t \)) occupying position \( X \). The velocity field would be written as

\[
v = v(x, t)
\]

but the velocity is still the velocity of a particle \( i.e., \)

\[
v = \left( \frac{\partial x}{\partial t} \right)_x = \frac{Dx}{Dt}
\]  

where we have used a capital \( D \) to emphasize that this is a material derivative.

The particle acceleration is

\[
\left( \frac{\partial v}{\partial t} \right)_x = \frac{Dv}{Dt} = \frac{D^2x}{Dt^2}
\]

This is not the same as the local time derivative which is

\[
\left( \frac{\partial v}{\partial t} \right)_x
\]

We can find a relationship connecting these two derivatives using the following formula from calculus:

\[
\left( \frac{\partial a}{\partial y} \right)_z = \left( \frac{\partial a}{\partial y} \right)_p + \left( \frac{\partial a}{\partial p} \right)_y \left( \frac{\partial p}{\partial y} \right)_z
\]  

so

\[
\left( \frac{\partial v}{\partial t} \right)_x = \left( \frac{\partial v}{\partial t} \right)_x + \left( \frac{\partial v}{\partial x} \right)_t \left( \frac{\partial x}{\partial t} \right)_x
\]

or
\[ \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \cdot \nabla v \]  

(9)

where \( \nabla v \) denotes the gradient with respect to spatial coordinates. We can use a scalar field in equation 8 such as density, \( \rho(x, t) \) i.e.,

\[ \left( \frac{\partial \rho}{\partial t} \right)_x = \left( \frac{\partial \rho}{\partial t} \right)_x + \left( \frac{\partial \rho}{\partial x} \right)_t \left( \frac{\partial x}{\partial t} \right)_x \]

or

\[ \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho \]

(10)

and, in general, we have the following relationship between the material and the local derivative:

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla \]

(11)

4. Conservation laws

We shall be using Gauss’ theorem quite a lot in the following i.e.,

\[ \int_S v \cdot \hat{n} \, dS = \int_V \nabla \cdot v \, dV \]

(12)

where \( v \) is a vector. More generally we have that

\[ \int_S \hat{n} \ast A \, dS = \int_V \nabla \ast A \, dV \]

(13)

where \( A \) may be a scalar, vector or tensor and \( \ast \) can be an ordinary product, vector dot product or vector cross product depending upon the context. We also note that the \textit{flux} of \( A \) through a surface \( S \) is given by

\[ \int_S \rho A v \cdot \hat{n} \, dS \]

(14)

(sometimes \( \hat{n} \, dS \) is denoted by \( dA \)).

With these preliminaries out of the way, we now turn to \textit{conservation of mass}. The mass of a volume \( V \) is given by

\[ M = \int_V \rho \, dV \]

so the rate of increase of \( M \) is given by

\[ \frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t} \, dV \]

provided that the surface of \( V \) is fixed in space. As we hypothesize no creation or destruction of mass, it follows that \( \frac{\partial M}{\partial t} \) must equal the rate of inflow of mass. From 14 and 12, the rate of inflow of mass is given by

\[ -\int_S \rho v \cdot \hat{n} \, dS = -\int_V \nabla \cdot (\rho v) \, dV \]
where the minus sign arises as we are considering an *inward* flux of material. Thus

\[
\int_V \frac{\partial \rho}{\partial t} \, dV = - \int_V \nabla \cdot (\rho \mathbf{v}) \, dV
\]

therefore

\[
\int_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \, dV = 0
\]

and, because this is true for an arbitrary volume, we have

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{15}
\]

or, equivalently

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{16}
\]

These last two equations are both statements of the conservation of mass. Note that if \( \nabla \cdot \mathbf{v} = 0 \), it immediately follows that \( D\rho/Dt = 0 \) so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics.

The most useful conservation law we shall use is the *conservation of linear momentum* but to derive it we shall need the *Reynolds mass transport theorem* which is proved in Malvern (1969). This theorem states that

\[
\frac{D}{Dt} \int_V \rho A \, dV = \int_V \rho \frac{D\mathbf{A}}{Dt} \, dV \tag{17}
\]

where \( A \) can be a vector, scalar or tensor.

The conservation of linear momentum is a basic postulate of continuum mechanics and can be stated as: the time rate of change of total momentum of a given set of particles equals the vector sum of all external forces acting on the particles. In mathematical form we have

\[
\frac{D}{Dt} \int_V \rho \mathbf{v} \, dV = \int_S \mathbf{t} \, dS + \int_V \rho \mathbf{b} \, dV \tag{18}
\]

or by 17

\[
\int_V \rho \frac{D\mathbf{v}}{Dt} \, dV = \int_S \mathbf{t} \, dS + \int_V \rho \mathbf{b} \, dV \tag{19}
\]

Now \( \mathbf{t} = \hat{n} \cdot \mathbf{T} \) so Gauss’ theorem can be used to convert the surface integral into a volume integral *i.e.*,

\[
\int_S \mathbf{t} \, dS = \int_S \hat{n} \cdot \mathbf{T} \, dS = \int_V \nabla \cdot \mathbf{T} \, dV
\]

Combining this with the previous equation gives

\[
\int_V \left( \rho \frac{D\mathbf{v}}{Dt} - \nabla \cdot \mathbf{T} - \rho \mathbf{b} \right) \, dV = 0
\]

which must hold for an arbitrary volume so

\[
\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{20}
\]
These are Cauchy’s equations of motion and they apply to the current deformed configuration. We have not made any approximation about the constitutive relationship or the size of the deformation.

5. Equilibrium conditions inside the Earth

The equilibrium stress state inside the Earth is mainly due to the pressure of overburden. On long time scales, the Earth behaves like a fluid and we often approximate the stress state as one of hydrostatic pressure. Such a stress field is isotropic. The deviatoric initial stresses (due to lateral density variations caused dominantly by convective motions) are almost certainly very small and can probably be neglected. In equilibrium, \( \nu = 0 \) so (20) becomes

\[
\nabla \cdot \mathbf{T}_0 + \rho_0 \mathbf{b} = 0
\]

where the zero subscript refers to the equilibrium state. The body force here is gravity \( i.e., \)

\[
\mathbf{b} \equiv \mathbf{g} = -\nabla \phi_0
\]

where \( \phi_0 \) is the gravitational potential and satisfies Poisson’s equation \( i.e., \)

\[
\nabla^2 \phi_0 = 4\pi G \rho_0
\]

We now have

\[
\nabla \cdot \mathbf{T}_0 - \rho_0 \nabla \phi_0 = 0
\]

Let \( \mathbf{T}_0 \) consist only of a hydrostatic pressure and assume there is no deviatoric part, then

\[
\mathbf{T}_0 = -p_0 \mathbf{I}
\]

so

\[
-\nabla p_0 - \rho_0 \nabla \phi_0 = 0
\]

We often assume that \( \rho_0 \) is a function of radius, \( r \) alone (in which case it follows that \( \phi_0 \) and \( p_0 \) are functions of radius alone – show this using (25)). If we are given \( \rho_0(r) \), it is trivial to compute \( \mathbf{g}(r) \) and \( p_0(r) \). Note that \( \mathbf{g} \) points inwards so it is convenient to write

\[
\mathbf{g}(r) = -\hat{r} \mathbf{g}_0(r)
\]

where

\[
\mathbf{g}_0(r) = \frac{\partial \phi_0}{\partial r}
\]

From Poisson’s equation with spherical symmetry

\[
\nabla^2 \phi_0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi_0}{\partial r} \right) = 4\pi G \rho_0
\]

or

\[
\left( \frac{\partial}{\partial r} r^2 \phi_0 \right) = 4\pi G \rho_0 r^2
\]

so

\[
\phi_0(r) = \frac{1}{r^2} \int_0^r 4\pi G \rho_0 x^2 \, dx
\]

5
Now from 25

\[ \hat{r} \frac{\partial p_0}{\partial r} = -\hat{r} \rho_0 \frac{\partial \phi_0}{\partial r} \]

so

\[ \frac{\partial p_0}{\partial r} = -\rho_0 g_0 \]  \hspace{1cm} (29)

Thus, given \( \rho_0(r) \), 28 can be evaluated to give \( g_0(r) \) and 29 can be integrated to give \( p_0(r) \) with the boundary condition that the equilibrium pressure at the surface of the Earth is zero (fig 2).

Figure 2. Pressure, gravity, and density in the Earth.

Because of the fact that we integrate the density profile to get pressure, the pressure distribution in the Earth is very precisely known and varies little from one density model to another. This means that mineral physicists can rely on the seismologically determined pressure at the 660km discontinuity (say) which is important for determining the temperature in the deep Earth.

The magnitude of non-hydrostatic stresses can be estimated in a variety of ways. They are probably greatest in the brittle crust and the stress drop during earthquakes can be used as a measure of the ambient stress levels. Stress drops are typically less than a few hundred bars (1 bar is approximately equal to 1 atmosphere and is equal to \( 10^5 \) pascals where the pascal is the SI unit of pressure). Pressures inside the deep Earth are of the order of Mbars (or several hundred gigapascals (Gpa)) so we may safely ignore non-hydrostatic stresses except perhaps near the surface.

6. Properties of convecting regions

The properties of the lower mantle and outer core appear to be consistent with those of a well-mixed, vigorously convecting region. How do we know this? Consider the likely thermal and chemical state of a vigorously convecting region. Convection is a very good mixer so that a convecting region tends to be homogeneous. The temperature is highly variable laterally. If the region is heated from below, boundary layers develop at the top and bottom of the region. These become unstable and plumes detach from both boundaries at irregular intervals. If we average across horizontal surfaces, we find that the interior
distribution is adiabatic. (The situation is a little more complicated if the heating is internal). If we sketch the horizontally averaged temperature profile we find:

The interior is adiabatic because the time scale of convective heat transport is much shorter than the time scale of diffusive processes such as conduction. The time scale of conductive heat transport is

\[ \tau_{\text{cond}} \approx \frac{\ell^2}{\kappa} \]

where \( \kappa \) is the thermal diffusivity given by:

\[ \kappa = \frac{k}{\rho C_p} \]

where \( k \) is thermal conductivity, \( \rho \) is density, and \( C_p \) is the heat capacity. \( \ell \) is a characteristic length scale. The time scale of conductive heat transport is

\[ \tau_{\text{conv}} \approx \frac{\ell u}{\kappa} \]

where \( u \) is a characteristic velocity in the convecting region. The ratio of these two time scales is called the Peclet number:

\[ \text{Pe} = \frac{\tau_{\text{cond}}}{\tau_{\text{conv}}} = \frac{\ell u}{\kappa} \]

With \( \kappa \approx 10^{-6} \text{ m}^2/\text{s}, \ell \approx 10^6 \text{m} \) and \( u \approx 5 \times 10^{-2} \text{ m/yr} \) we find that \( \text{Pe} \approx 1500 \). This means that conductive time scales are three orders of magnitude larger than convective time scales.

If we completely ignore diffusion processes, a parcel of fluid moving in a pressure field is adiabatically compressed as it moves down and is adiabatically decompressed as it moves up. In the body of the convecting region, the radial gradient of the laterally averaged entropy, \( s \) is zero. In a vigorously convecting region, we therefore expect a state of homogeneity and adiabaticity. If this is true, we can calculate what the expected density distribution would be using the seismically determined elastic moduli. We can then compare this with seismically determined density and see if the properties are really consistent with vigorous convection.

The density variation in a homogeneous and adiabatic region is controlled by the adiabatic compressibility, \( \beta_s \), defined as

\[ \beta_s = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial p} \right)_s \]

(the adiabatic bulk modulus \( K_s \) is just \( 1/\beta_s \)). Obviously

\[ \left( \frac{d\rho}{dr} \right)_{ad} = \left( \frac{\partial \rho}{\partial p} \right)_s \frac{dp}{dr} = -\rho g \left( \frac{\partial p}{\partial p} \right)_s \]
\( \frac{\partial p}{\partial \rho} \) is often given the symbol \( \phi \) and is called the “seismic parameter” because it can be computed from the velocities of propagation of seismic waves in the Earth:

\[
\phi = V_p^2 - \frac{4}{3} V_s^2
\]

The equation \( (d\rho/dr)_{ad} = -g\rho/\phi \) is called the Adams-Williamson equation and is obeyed, within the limits of data resolution, in the lower-mantle and outer-core of the Earth (see below). This equation was extensively used in the 1950’s to construct models of the density as a function of radius in the Earth. Later models constrained \( \rho(r) \) with measurements of the periods of the free oscillations of the Earth.

7. Temperature distribution in a convecting region

We start with the fact that we are dealing with an isentropic region (when laterally averaged):

\[
\frac{ds}{dr} = 0
\]

In a homogeneous medium, the entropy is a function of temperature and pressure so we can write

\[
\frac{ds}{dr} = \left( \frac{\partial s}{\partial T} \right)_p \frac{dT}{dr} + \left( \frac{\partial s}{\partial p} \right)_T \frac{dp}{dr} = 0
\]

The specific heat, \( C_p \), is defined as

\[
C_p = T \left( \frac{\partial s}{\partial T} \right)_p
\]

and using one of Maxwell’s relations we have

\[
\left( \frac{\partial s}{\partial p} \right)_T = \frac{1}{\rho^2} \left( \frac{\partial \rho}{\partial T} \right)_p = -\frac{\alpha}{\rho}
\]

so

\[
\frac{C_p}{T} \frac{dT}{dr} - \frac{\alpha}{\rho} \frac{dp}{dr} = 0
\]

Hence

\[
\left( \frac{dT}{dr} \right)_{ad} = \frac{\alpha T}{\rho C_p} \frac{dp}{dr} = -\frac{\alpha g T}{C_p}
\]

If we know \( \alpha, C_p \) and a boundary condition on \( T \) we can integrate this expression to get the temperature profile in the body of a convecting region.

The equations for \( (dT/dr)_{ad} \) is only valid in a homogeneous region as we have omitted the dependence of entropy and density on the phase and composition of the material. Additional terms must therefore be considered when computing the spherically averaged temperature profile in the upper mantle.

When computing the temperature profile in the deep Earth it is convenient to introduce a dimensionless number called Grüneisen’s ratio. We have

\[
\left( \frac{\partial T}{\partial r} \right)_{ad} = -\frac{\alpha g T}{C_p} = \left( \frac{\partial T}{\partial p} \right)_s \frac{dp}{dr}
\]

so

\[
\left( \frac{\partial T}{\partial p} \right)_s = \frac{\alpha T}{\rho C_p}
\]

It is also convenient to cast the problem in terms of the variation of temperature with density:
\[
\left( \frac{\partial T}{\partial \rho} \right)_s = \left( \frac{\partial T}{\partial p} \right)_s \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{\alpha \phi T}{C_p \rho} = \frac{T}{\rho}
\]

where

\[
\gamma = \frac{\alpha \phi}{C_p}
\]

\(\gamma\) is Grüneisen’s ratio and falls in the range 1 – 2 for most materials in the Earth. It is also weakly dependent on \(T\) and \(\rho\) and, to a first approximation, can be considered a constant, \(i.e.,\)

\[
\left( \frac{\partial \ln T}{\partial \ln \rho} \right)_s = \gamma \quad \text{ (a constant)}
\]

If we integrate this equation we have

\[
\frac{T}{T_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma
\]

which is useful for giving an estimate of the temperature rise across the lower mantle or the outer core.