Mantle convection – continued

1. Boundary layer theory

For the purposes of this section, we consider incompressible Boussinesq flow in a 2d geometry. The equations then become: (mass)

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

(momentum)

$$\nabla \cdot \bar{\mathbf{T}}' = \nabla \bar{P}' + \alpha\rho\bar{\mathbf{g}}(T - T_A) \tag{2}$$

For an incompressible newtonian fluid with visocsity independent of position, this becomes

$$\eta\nabla^2\mathbf{v} = \nabla \bar{P}' + \alpha\rho\bar{\mathbf{g}}(T - T_A) \tag{3}$$

(energy) – this assumes that k is roughly independent of position

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa\nabla^2 T + \frac{h}{C_p} \tag{4}$$

We introduce the vorticity:

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{5}$$

Note that $\nabla^2\mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) = -\nabla \times \boldsymbol{\omega}$. Take the curl of the momentum equation (to eliminate pressure which is proportional to the gradient of a scalar) giving

$$\eta\nabla^2\boldsymbol{\omega} = \rho\alpha\nabla T \times \mathbf{g} \tag{6}$$

Since $\mathbf{g}$ effectively acts in the z direction, $\mathbf{g} = -g\hat{\mathbf{z}}$ then

$$\nabla^2\omega = \frac{\rho\alpha g}{\eta}\frac{\partial T}{\partial x} \tag{7}$$

where ω is the y component of $\boldsymbol{\omega}$. At sufficiently high Rayleigh number (though provided that the flow is reasonably steady state) analytical approximations to the convective solution can be made. The flow is characterized by thin boundary layers, thin plumes, and an isothermal interior. Consider the energy equation. For no internal heating and steady state, this reduces to

$$\mathbf{v} \cdot \nabla T = \kappa\nabla^2 T \tag{8}$$

In the thin boundary layer, $V_z \simeq 0$ and, away from the rising and descending plumes, the isotherms are nearly horizontal so conduction in the z direction is the most important. Thus, in boundary layers

$$v_x\frac{\partial T}{\partial x} \simeq \kappa\frac{\partial^2 T}{\partial z^2} \tag{9}$$

i.e., we have a balance between horizontal advection of heat and vertical diffusion of heat. In rising and descending plumes, $v_x \simeq 0$ and $\partial T/\partial z \simeq 0$ so the energy equation isn't much use. Instead we use the vorticity equation above. In 2d,

$$\omega = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \tag{10}$$

so, in a plume, $v_x \simeq 0$ so $\omega \simeq -\partial v_z/\partial x$. At the edge of the box (in the middle of a plume) $\partial v_z/\partial x \simeq 0$ so $\omega \simeq 0$ thus ω rapidly varies from the interior value to zero at the edge and is mainly a function of x:

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\rho\alpha g}{\eta}\frac{\partial T}{\partial x} \tag{11}$$

in plumes. The interior of the flow is nearly isothermal so the equation reduces to

$$\nabla^2 \omega = 0 \tag{12}$$

in the interior. In fact, with no internal heating, the interior of the flow is basically a steady rotation with constant ω.

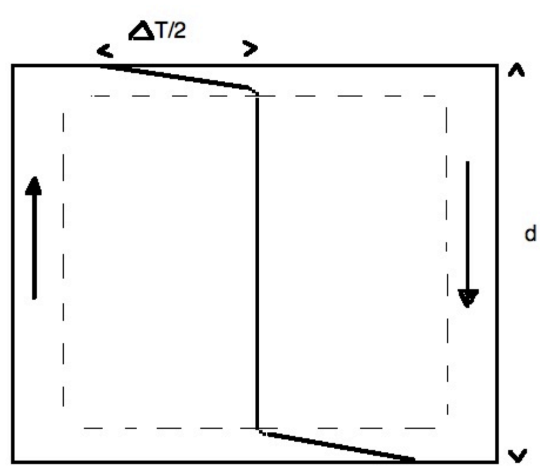
We can now make a simple dimensional analysis to determine the functional forms for the boundary layer thickness, flow speed in boundary layers, and the Nusselt number. These results can then be made quantitative using the results of numerical calculations and laboratory experiments. We again consider the simplest case of applying a fixed temperature difference (ΔT) across a box (no internal heating). In the absence of flow, the energy equation reduces to

$$\kappa \frac{\partial^2 T}{\partial z^2} = 0 \tag{13}$$

and the solution for temperature is linear with temperature gradient $\partial T/\partial z = \Delta T/d$ where d is the thickness of the layer. The conducted heat flux out of the surface of the box is therefore

$$|q| = \frac{k\Delta T}{d} \tag{14}$$

When we have convection, the temperature profile away from plumes looks like :



Let δ be the thickness of the boundary layer so that, in the middle of the box,

$$\frac{\partial^2 T}{\partial z^2} \simeq \frac{\Delta T}{2\delta^2} \tag{15}$$

as $\partial T/\partial z$ changes from zero to $\Delta T/2\delta$ across the boundary layer. The horizontal temperature gradient across the boundary layer will be $\partial T/\partial x \simeq \Delta T/d$ thus we can estimate a mean value of v_x in the boundary layer. Since

$$v_x \frac{\partial T}{\partial x} \simeq \kappa \frac{\partial^2 T}{\partial z^2} \tag{16}$$

we get

$$v_x \simeq \frac{\kappa d}{2\delta^2} \tag{17}$$

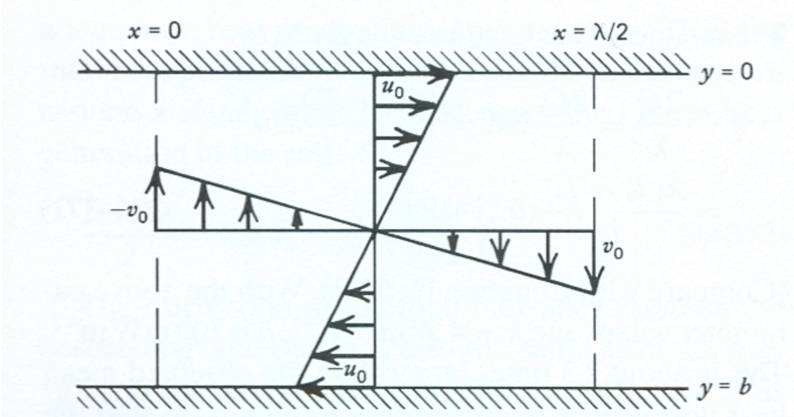
Similarly, in a plume we have (since $\partial T/\partial x$ across the middle of the plume is approximately $\Delta T/2\delta$)

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{\rho\alpha g}{\eta}\frac{\partial T}{\partial x} \simeq \frac{\rho\alpha g \Delta T}{2\eta\delta} \tag{18}$$

ω changes from a nearly constant interior value to zero across the plume so $\partial^2\omega/\partial x^2 \simeq \bar{\omega}/\delta^2$ where $\bar{\omega}$ is the interior value giving

$$\bar{\omega} \simeq \frac{\rho\alpha g \Delta T \delta}{2\eta} \tag{19}$$

If we now consider the interior of the flow, we have roughly



6–41 Linear velocity profiles used to model the core flow in a convection cell. The areas under the triangles are equal to conserve fluid.

In a horizontal slice across the middle, we have

$$\bar{\omega} = -\frac{\partial v_z}{\partial x} \simeq \frac{2v_z}{d} \tag{20}$$

where v_z is the mean vertical velocity at the edge of the plume. Similarly, if we consider a vertical slice across the middle, we have

$$\bar{\omega} \simeq \frac{2v_x}{d} \tag{21}$$

where v_x is the mean horizontal velocity at the edge of the boundary layer. Thus $v_x \simeq v_z$ (at least for a square box) which isn't surprising as we have to conserve mass. let $v \simeq v_x \simeq v_z$ and we have

$$v = \frac{\kappa d}{2\delta^2} \quad \text{and} \quad \frac{2v}{d} = \frac{\rho\alpha g \Delta T \delta}{2\eta} \tag{22}$$

The Rayleigh number for this system is given by $Ra = \rho\alpha g \Delta T d^3/\kappa\eta$ and combining equations appropriately gives

$$\delta \propto d Ra^{-1/3} \tag{23}$$

$$v \propto \frac{\kappa}{d} Ra^{2/3} \tag{24}$$

The heat flow out of the surface of the box when we have convection is

$$|q| = \frac{k\Delta T}{2\delta} \tag{25}$$

Thus the Nusselt number is

$$\frac{k\Delta T}{2\delta} / \frac{k\Delta T}{d} = \frac{d}{2\delta} \propto Ra^{1/3} \tag{26}$$

Numerical calculations verify this relationship for Rayleigh numbers at least a factor of 10 above critical and the relation is extremely useful for calculating thermal histories of the earth. We also note that this boundary layer analysis can also be developed for internally heated flows where slightly different results are obtained.

2. Convective efficiency

A treatment of the global entropy and energy equations leads to the concept of convective efficiency. The energy equation can be manipulated to give the desired result. For simplicity, we assume a steady state and that the boundaries of the mantle are not moving radially ($\mathbf{v} \cdot \hat{\mathbf{z}} = 0$). A steady state implies that $\partial\rho/\partial t = 0$ so conservation of mass gives $\nabla \cdot (\rho \mathbf{v}) = 0$. Integrating the energy equation over the whole mantle gives the rather obvious result (for steady state)

$$Q = \int_S \mathbf{q} \cdot \mathbf{dS} = \int_V \rho h dV \tag{27}$$

where Q (the net heat flux out of the mantle) is just balanced by internal radioactive heat production. To look at dissipation, we must use the entropy equation

$$\rho T \frac{Ds}{Dt} = \rho C_p \left[\frac{DT}{Dt} - \left(\frac{\partial T}{\partial P} \right)_S \frac{DP}{Dt} \right] = \rho h - \nabla \cdot \mathbf{q} + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{28}$$

Integrating this equation over the mantle gives (assuming steady state)

$$\int_V \rho C_p \mathbf{v} \cdot \nabla T dV - \int_V \alpha T \mathbf{v} \cdot \nabla P dV + \int_S \mathbf{q} \cdot \mathbf{dS} - \int_V \rho h dV = \int_V \mathbf{T}' : \dot{\boldsymbol{\epsilon}} dV = \Phi \tag{29}$$

where Φ is the global rate of viscous heating. To a good approximation, C_p is a constant in the mantle then we can write

$$\int_V \rho C_p \mathbf{v} \cdot \nabla T dV = \int_V C_p \nabla \cdot (\rho \mathbf{v} T) dV = \int_S \rho T C_p \mathbf{v} \cdot \mathbf{dS} = 0 \tag{30}$$

where we have assumed the mantle is neither expanding or contracting and we have used $\nabla \cdot (\rho \mathbf{v}) = 0$. If we now use the global conservation of energy, we have

$$\Phi = -\int_V \alpha T \mathbf{v} \cdot \nabla P dV \tag{31}$$

This equation implies that the global rate of dissipative heating is exactly cancelled by the work done against the adiabatic gradient. It turns out that the pressure gradient is dominated by the hydrostatic background term so that $\nabla P \simeq -\rho g \hat{\mathbf{z}}$ where $\hat{\mathbf{z}}$ points in the upward vertical direction. Then

$$\Phi = \int_V \frac{g\alpha}{C_p} \rho C_p T V_z dV \tag{32}$$

where V_z is the vertical velocity. If we average over horizontal surfaces, we see that $< \rho C_p T V_z >$ is the horizontally averaged convective heat flux, and using the definition of the Dissipation number (assumed constant) we find that

$$\Phi = Di Q_{conv} \leq Di Q_s \tag{33}$$

where Q_s is the total heat flux out of the top surface and Q_{conv} is the convected heat flux. When the convection is vigorous (i.e. large Rayleigh number), the Nusselt number is large and $Q_{conv} \simeq Q_s$ so we can define an "efficiency" as

$$\frac{\Phi}{Q_s} \leq Di \tag{34}$$

Clearly, when the dissipation number is small and the Boussinesq approximation is valid, the global rate of viscous dissipation is small and can be neglected. The dissipation number depends on the depth scale of convection and for whole-mantle convection, $Di \simeq 0.5$ so that it is possible that viscous dissipation is important on a global scale.

For internal heating, the efficiency must be modified a bit as the convective heat flux now changes as a function of depth and we find that viscous dissipation can be reduced.

This global analysis says nothing about the local importance of dissipative heating. Some calculations are presented by Tackley and are shown in the accompanying ppt.

3. Thermal history

The Nusselt number - Rayleigh number relationship derived above and verified in numerical experiments on isoviscous fluids can be used in thermal history calculations:

$$Nu = aRa^{\beta} \tag{35}$$

with $\beta \simeq 0.3$. One might think that assuming a constant viscosity is not a particularly good approximation for the Earth and it is certainly true that including a temperature dependent viscosity can dramatically change the exponent. Typically what happens is that the cold upper boundary layer becomes highly viscous and forms a "stagnant lid" which strongly reduces the efficiency of convective heat flow and β can become 0.1 or smaller. Such a relationship would probably be appropriate to use on Venus but Earth has plate tectonics and the upper boundary layer gets recycled in a fashion more similar to the isoviscous calculations.

To do thermal history calculations, we use the energy equation integrated over the volume of the mantle. The mantle is allowed to cool and, in most cases, the adiabatic heating term and viscous heating terms are neglected. As shown above, these terms cancel globally if the mantle neither expands or contracts though this is unlikely to be a good approximation over the age of the Earth. We write the simplified energy equation as

$$\int_V \rho C_p \frac{\partial T}{\partial t} dV = -Q_{out} + Q_{in} + \int_V \rho h dV \tag{36}$$

We now write $d\bar{T}/dt$ as the mean cooling rate of the mantle giving

$$MC_p \frac{d\bar{T}}{dt} = -Q_{out} + Q_{in} + E \tag{37}$$

where E is the radioactive heat generation in the mantle and M is the mass of the mantle. Note that E is a function of time and is roughly exponentially decreasing. We are going to use the form for the Nusselt number defined above but we need to be a little careful about our definition of Rayleigh number. The usual form is

$$Ra = \frac{\alpha g \rho \bar{T} d^3}{\kappa \eta} \tag{38}$$

where we have normalized the temperature such that the surface temperature is zero and $\bar{T}$ is the mean interior temperature. The Nusselt number can be written as

$$Nu = \frac{Q_{out}}{Q_{cond}(\bar{T})} \tag{39}$$

where $Q_{cond}(\bar{T})$ is the hypothetical heat flow that would emerge with the given average temperature, $\bar{T}$ and when only conduction operates. Generally, we can write $Q_{cond}(\bar{T}) = c\bar{T}$ so that

$$Q_{out} = ac\bar{T}Ra^{\beta} \tag{40}$$

Incorporating this with the definition of the Rayleigh number and treating everything as a constant except for η which may be a function of temperature and so will change with time, we have

$$MC_p \frac{d\bar{T}}{dt} = Q_{in}(t) + E(t) - a' \frac{\bar{T}^{1+\beta}}{[\eta(\bar{T})]^{\beta}} \tag{41}$$

We now have to specify a viscosity law, e.g.

$$\eta = \eta_0 \exp\left(\frac{gT_m}{\bar{T}}\right) \tag{42}$$

Alternatively, we can specify the viscosity relative to a reference temperature and write an approximate form:

$$\eta = \eta_0 \left(\frac{\bar{T}}{T_0}\right)^{-n} \tag{43}$$

The reference temperature might be the present mean temperature and, provided $\bar{T}$ doesn't change dramatically with time, this equation represents the exponential behavior reasonably well. Note that n is probably in the range 30 to 40.

The energy equation can now be integrated backward in time (using the present conditions as initial conditions) or forward in time using some guess of the initial conditions. The constant a' can be estimated from numerical calculations or it can be fudged out by normalizing Q_{out} to be some specific value at a particular value of $\bar{T}$. A natural choice is at $t = t_0$ (at present time) set Q_{out} to Q_0 and $\bar{T} = T_0$. Q_0 is the current day heat loss from the mantle and is thought to be about 80% of the total surface heat loss(after correction for continental heat production). Combining these results together gives

$$MC_p \frac{d\bar{T}}{dt} = Q_{in}(t) + E(t) - Q_0 \left(\frac{\bar{T}}{T_0}\right)^{1+\beta+n\beta} \tag{44}$$

This form is convenient as it allows the thermal response of the Earth to be analytically investigated for some simple cases. Consider the case when we have no heat sources:

$$MC_p \frac{d\bar{T}}{dt} = -Q_0 \left(\frac{\bar{T}}{T_0}\right)^{m} \tag{45}$$

where $m = 1 + \beta + n\beta$. For $n = 30 \rightarrow 40$ and $\beta \simeq .3$ we find that $m = 12$. If convective heat transport is ignored and conduction dominates ($\beta = 0$) then $m = 1$. (Actually m might be larger than 1 for conduction because of contributions of radiative heat transfer which would lead to a temperature dependent thermal conductivity and could give an m of 2 to 4.) When $m = 1$, the solution to the above equation is

$$\bar{T} = T_0 \exp\left[-\frac{Q_0}{T_0 MC_p}(t - t_0)\right] \tag{46}$$

which gives a conductive time scale of cooling of $T_0 MC_p/Q_0$ of approximate 8By. If m is greater than 1 then the solution looks like

$$\left(\frac{\bar{T}}{T_0}\right)^{m-1} = 1 + \frac{Q_0}{T_0 MC_p}(t - t_0)(m-1) \tag{47}$$

It is interesting that this equation can lead to infinite temperatures in the past – this happens in the last 4By if m is greater than 2. Both of these results suggest that the assumption of no internal heat sources is inconsistent with the present day heat flow.

When we have internal heat sources, it is possible to ask what the thermal response time is if, at some time, we increase stepwise the amount of heating. The solution has a decay constant τ where

$$\tau = \frac{MC_p T_0}{mQ_0} \tag{48}$$

which is the conductive time constant divided by m. For $m = 12$, $\tau \simeq$ 700my so there is time for thermal impulses to decay. For smaller values of β associated with stagnant lid convection, $\beta = .1$ so

$m \simeq 5$ and $\tau \simeq$1.5By which is a significant fraction of the age of the Earth. Numerical integration of the equations leads to some general results which are characterized by the "Urey ratio" which is the ratio of heat produced to heat lost as a function of time. The variation with time of heat production by radioactive elements is usually characterized by some average half life of the major heat producing elements assuming that their relative abundances has remained constant over geological time. With β in the range 0.2 to 0.35, we arrive at the following conclusions:

1) The internal heat production is severely constrained to prevent models ending up with $\bar{T}$ either going to infinity or to zero as we go back in time. The present Urey ratio is in the range 0.7 to 0.9 and stays constant over roughly the last 2.5By. This result is consistent with the self-regulation hypothesis where, for efficient convection, the Earth basically manages to lose all the heat that is produced and there is a rough balance between heat loss and heat production.
2) The mantle has forgotten its initial condition, i.e., it arrives at the same temperature after about 1By whether it has a cold or a hot origin
3) The temperature drop over the last 3By is between 150K and 250K
4) Heat flow was higher in the past so plate velocities were correspondingly higher. In a cooling plate model, heat flow is proportional to the square root of the spreading rate so, if the heat flow is 2 to 5 times higher in the past, spreading rates could have been 4 to 25 times higher.

There is some evidence that the mean temperature of the mantle was higher in the Archean. This comes from the presence of komatiites which are igneous rocks with a high MgO content and require temperatures which are 200K to 400K higher than present. On the other hand, Archean geotherms estimated from metamorphic mineral assemblages seem to be similar to the present day geotherm. There is also little evidence for plate rates being significantly higher than at present.