# Mantle convection: equations

This material is treated in detail in Chapter 6 of Schubert, Turcotte and Olsen: Mantle Convection in the Earth and Planets, CUP, 2001

## 1. Conservation of Mass

We derived this (and conservation of linear momentum) earlier in class

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

or, equivalently

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2}$$

Note that if $\nabla \cdot \mathbf{v} = 0$, it immediately follows that $D\rho/Dt = 0$ so that the density of a particle does not change with time. This states that the medium is *incompressible* and is a commonly used approximation in fluid mechanics.

## 2. Conservation of Linear Momentum

The total body force acting on a body is

$$\int_V \rho \mathbf{g} \, dV \tag{3}$$

and the total surface force acting on the body is

$$\int_S \mathbf{t} \, dS \tag{4}$$

where $\mathbf{t}$ is the traction vector and

$$\mathbf{t} = \hat{\mathbf{n}} \cdot \mathbf{T} \tag{5}$$

This equation defines the symmetric *Cauchy stress tensor*, $\mathbf{T}$, which is the linear vector function which associates with each unit normal $\hat{\mathbf{n}}$ the traction vector $\mathbf{t}$ acting at the point across the surface whose normal is $\hat{\mathbf{n}}$. The mean normal pressure is defined as

$$P = -\tfrac{1}{3} \left( T_{kk} \right) \tag{6}$$

Conservation of linear momentum can be written

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{g} \tag{7}$$

which are Cauchy's equations of motion and they apply to the current deformed configuration. We have not made any approximation about the constitutive relationship or the size of the deformation.

## 3. Conservation of energy

Body forces and surface forces do work on a parcel of fluid and change the internal energy and the kinetic energy. The rate at which work is done (the input power) is

$$P_{input} = \int\limits_S \mathbf{t} \cdot \mathbf{v} dS + \int\limits_V \rho \mathbf{g} \cdot \mathbf{v} dV \tag{8}$$

This can be separated into two contributions; mechanical work performed in deforming the body and work done in changing the kinetic energy of the body. The mathematical development is as follows (remember that $A \cdot\cdot B = A_{ij}B_{ji}$ and $A : B = A_{ij}B_{ij}$ in a Cartesian coordinate system):

$$\begin{aligned}
P_{input} &= \int\limits_S \hat{\mathbf{n}} \cdot \mathbf{T} \cdot \mathbf{v}\, dS + \int\limits_V \rho \mathbf{g} \cdot \mathbf{v}\, dV \\
&= \int\limits_V [\nabla \cdot (\mathbf{T} \cdot \mathbf{v}) + \rho \mathbf{g} \cdot \mathbf{v}]\, dV \qquad \text{(using Gauss' theorem)} \\
&= \int\limits_V [(\nabla \cdot \mathbf{T}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v} + \rho \mathbf{g} \cdot \mathbf{v}]\, dV \\
&= \int\limits_V [(\nabla \cdot \mathbf{T} + \rho \mathbf{g}) \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}]\, dV \\
&= \int\limits_V [\rho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} + \mathbf{T} \cdot\cdot \nabla \mathbf{v}]\, dV \\
&= \int\limits_V \tfrac{1}{2}\, \rho \frac{D}{Dt}(\mathbf{v} \cdot \mathbf{v})\, dV + \int\limits_V \mathbf{T} \cdot\cdot \nabla \mathbf{v}\, dV \\
&= \frac{D}{Dt} \int\limits_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int\limits_V \mathbf{T} \cdot\cdot \nabla \mathbf{v}\, dV \\
&= \frac{D}{Dt} \int\limits_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int\limits_V \mathbf{T} \cdot\cdot (\dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\Omega}})\, dV \\
&= \frac{D}{Dt} \int\limits_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int\limits_V \mathbf{T} \cdot\cdot \dot{\boldsymbol{\epsilon}}\, dV \qquad \text{because } \mathbf{T} \text{ is symmetric and } \boldsymbol{\Omega} \text{ is antisymmetric}
\end{aligned}$$

Note that we separated the gradient of velocity tensor into a symetric and antisymmetric part: $\nabla \mathbf{v} = \dot{\boldsymbol{\epsilon}} + \dot{\boldsymbol{\Omega}}$) where $\dot{\boldsymbol{\epsilon}}$ is the symmetric strain rate tensor. Finally we get

$$P_{input} = \frac{D}{Dt} \int\limits_V \tfrac{1}{2}\, \rho \mathbf{v} \cdot \mathbf{v}\, dV + \int\limits_V \mathbf{T} : \dot{\boldsymbol{\epsilon}}\, dV \qquad \text{because } \mathbf{T} \text{ is symmetric} \tag{9}$$

which clearly shows the separation into mechanical work and kinetic energy. The mechanical work contributes to the change in internal energy (from the first law of thermodynamics).

There will be other sources of energy input which we denote $Q_{input}$. In mantle convection, we will have conduction and radioactive heat generation:

$$Q_{input} = -\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,dS + \int_V \rho h\,dV$$

where $h$ is the rate of heat generation per unit mass, $\mathbf{q} = -k\nabla T$ is the heat flux and $T$ is temperature. If $U$ is the total energy of the volume then

$$\dot{U} = P_{input} + Q_{input} \tag{10}$$

This is a statement of the first law of thermodynamics. $U$ consists of the change of kinetic energy plus the change of internal energy of the volume so

$$\dot{U} = \frac{D}{Dt}\int_V [\tfrac{1}{2}\,\rho\mathbf{v}\cdot\mathbf{v} + \rho E]\,dV$$

where $E$ is the internal energy per unit mass. As we have already separated out the change of kinetic energy in $P_{input}$ we can get an expression for the change of internal energy. Combining 9 and 10 gives

$$\frac{D}{Dt}\int_V \rho E\,dV = \int_V \mathbf{T}:\dot{\boldsymbol{\epsilon}}\,dV - \int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,dS + \int_V \rho h\,dV$$

or (using Gauss' theorem and the Reynolds mass transport theorem)

$$\rho\frac{DE}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}:\dot{\boldsymbol{\epsilon}} \tag{11}$$

For a homogeneous material, $dE = TdS - PdV = TdS + Pd\rho/\rho^2$ and

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP = \frac{C_p}{T}dT - \frac{\alpha}{\rho}dP$$

so

$$\rho\frac{DE}{Dt} = \rho C_p\frac{DT}{Dt} - \alpha T\frac{DP}{Dt} + \frac{P}{\rho}\frac{D\rho}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}:\dot{\boldsymbol{\epsilon}}$$

Finally we write $\mathbf{T}$ as a deviatoric and isotropic part (where the isotropic part is, by definition, the pressure)

$$\mathbf{T} = \mathbf{T}' - P\mathbf{I} \quad \text{whence} \quad \mathbf{T}:\dot{\boldsymbol{\epsilon}} = \mathbf{T}':\dot{\boldsymbol{\epsilon}} - P\nabla\cdot\mathbf{v} = \mathbf{T}':\dot{\boldsymbol{\epsilon}} + \frac{P}{\rho}\frac{D\rho}{Dt}$$

and so we end up with

$$\rho C_p\frac{DT}{Dt} - \alpha T\frac{DP}{Dt} = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}':\dot{\boldsymbol{\epsilon}} \tag{12}$$

or equivalently

$$\rho C_p\left[\frac{DT}{Dt} - \left(\frac{\partial T}{\partial P}\right)_S\frac{DP}{Dt}\right] = \rho h - \nabla\cdot\mathbf{q} + \mathbf{T}':\dot{\boldsymbol{\epsilon}} \tag{13}$$

The term in square brackets is the change in temperature in excess of that which comes from adiabatic compression.

## 4. Background reference state

We assume an adiabatic and hydrostatic reference state: $P_a, T_A, \rho_A$

$$\left.\begin{aligned} \nabla P_A &= \rho_A \mathbf{g} \\ \nabla \rho_a &= \nabla P_A \left(\frac{\partial \rho}{\partial P}\right)_S = \nabla P_A \frac{\rho}{K_s} \\ \nabla T_A &= \nabla P_A \left(\frac{\partial T}{\partial P}\right)_S = \nabla P_A \frac{\alpha T}{\rho C p} \end{aligned}\right\} \tag{14}$$

We will have (not necessarily small) departures from this reference state: $P = P_A + P'; \rho = \rho_A + \rho'; T = T_A + \theta$. We relate the perturbation in density $\rho'$ to perturbations in pressure and temperature using the following approximate equation of state:

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right)_T dP + \left(\frac{\partial \rho}{\partial T}\right)_P dT$$

so

$$\rho' = \frac{\rho_A}{K_T} P' - \alpha \rho_A \theta \simeq \frac{\rho_A}{K_S} P' - \alpha \rho_A \theta \tag{15}$$

(remember $K_S/K_T = 1 + \alpha T \gamma$ where the last term is on the order of 0.05).

## 5. Constitutive relation

$$T'_{ij} = 2\eta[\dot{\epsilon}_{ij} - \tfrac{1}{3}\,\dot{\epsilon}_{kk}\delta_{ij}] \tag{16}$$

where $\eta$ is the effective viscosity which can be a function of stress, temperature, etc. The term in square brackets is called the "strain rate deviator" and we use this because the deviatoric stress is trace-free by definition. It is conventional to ignore "bulk" viscosity which would arise during compression and expansion of a material.

## 6. The anelastic liquid approximation

The equations we have are general enough to include sound waves. One way of removing these is to set $\partial\rho/\partial t$ to zero in the conservation of mass equation. This is valid when the mach number is small. The mach number is a characteristic velocity of flow divided by the velocity of sound – for mantle convection, this number is tiny (on the order of $10^{-15}$. From equation (1) we get that

$$\nabla \cdot (\rho \mathbf{v}) = 0 = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$$

Using the reference state, mass conservation becomes:

$$\nabla \cdot \mathbf{v} + \frac{\alpha}{\gamma C_p} \mathbf{v} \cdot \mathbf{g} = 0 \tag{17}$$

The conservation of linear momentum becomes

$$\left.\begin{aligned}\rho\frac{D\mathbf{v}}{Dt} &= \nabla\cdot\mathbf{T}' - \nabla P + \rho\mathbf{g} \\ &= \nabla\cdot\mathbf{T}' - \nabla P' + \rho'\mathbf{g} \\ &= \nabla\cdot\mathbf{T}' - \nabla P' + \frac{\alpha\mathbf{g}P'}{C_p\gamma} - \alpha\rho_A\mathbf{g}\theta\end{aligned}\right\} \tag{18}$$

The conservation of energy requires a little more work. From equation 15 we have

$$\frac{DP}{Dt} = \frac{K_T}{\rho}\frac{D\rho}{Dt} + \alpha K_T\frac{DT}{Dt}$$

so

$$\alpha T\frac{DP}{Dt} = TC_v\gamma\frac{D\rho}{Dt} + \rho C_v\alpha T\gamma\frac{DT}{Dt}$$

so (using equation 1 and equation 17)

$$\alpha T\frac{DP}{Dt} = \frac{\alpha T\mathbf{v}\cdot\nabla P_A}{1+\alpha T\gamma} + \frac{\alpha T\gamma}{1+\alpha T\gamma}\rho C_p\frac{DT}{Dt}$$

Substituting into equation 12 and bearing in mind that $\alpha T\gamma$ is much less than 1 (about 0.05) gives

$$\rho C_p\frac{DT}{Dt} - \alpha T\mathbf{v}\cdot\nabla P_A = \rho_A h - \nabla\cdot[k\nabla(T_A+\theta)] + \mathbf{T}' : \dot{\boldsymbol{\epsilon}}$$

and because $\alpha T\mathbf{v}\cdot\nabla P_A = \rho C_p\mathbf{v}\cdot\nabla T_A$ we get

$$\rho C_p\frac{D\theta}{Dt} - \alpha\theta\mathbf{v}\cdot\nabla P_A = \rho_A h - \nabla\cdot[k\nabla(T_A+\theta)] + \mathbf{T}' : \dot{\boldsymbol{\epsilon}} \tag{19}$$

## 7. Scaling the equations

It is conventional to scale the equations (17–19) using the diffusion time scale $D^2/\kappa_0$ where $D$ is a charcteristic length scale (e.g. the depth of the mantle) and $\kappa_0 = k/\rho C_p$ is a reference thermal diffusivity. Velocity scales as $\kappa_0/D$ and stress scales as $\eta_0\kappa_0/D^2$ where $\eta_0$ is a reference viscosity. Temperatures can be scaled in a variety of ways, with internal heating we can use $\rho_0 h_0 D^2/k_0$ where $\rho_0$ is a reference density and $k_0$ is a reference thermal conductivity. If we define the "Dissipation number" $Di = \alpha g_0 D/C_p$ where $g_0$ is a reference value for gravity, we can write the equations as

$$\nabla\cdot\bar{\mathbf{v}} + \frac{Di}{\gamma}\bar{\mathbf{v}}\cdot\bar{\mathbf{g}} = 0 \tag{20}$$

$$\frac{\bar{\rho}_A}{Pr}\frac{D\bar{\mathbf{v}}}{D\bar{t}} = \nabla\cdot\bar{\mathbf{T}}' - \nabla\bar{P}' + \frac{Di}{\gamma}\bar{\rho}_A\bar{\mathbf{g}}\bar{P}' - Ra\bar{\rho}_A\bar{\mathbf{g}}\bar{\theta} \tag{21}$$

where $Pr$ is the Prandtl number: $Pr = \eta_0/\rho_0\kappa_0 = 10^{23}$. Clearly the first term can be neglected: inertial terms are irrelevant which means that if we stopped driving mantle convection the motion would stop instantaneously. $Ra$ is the Rayleigh number (see below). The energy equation becomes

$$\bar{\rho}_A\frac{D\bar{\theta}}{Dt} - Di\bar{\rho}_a\bar{\mathbf{g}}\cdot\bar{\mathbf{v}}\bar{\theta} = \nabla\cdot[\bar{k}(\nabla\bar{\theta} + Di\bar{T}_A)] + \frac{Di}{Ra}\bar{\mathbf{T}}'\cdot\cdot\nabla\bar{\mathbf{v}} + \bar{\rho}_A\bar{h} \tag{22}$$

where

$$Ra = \frac{\alpha_0 \rho_0^2 g_0 h_0 D^5}{k_0 \eta_0 \kappa_0}$$

The Boussinesq approximation (where the only place departures from the reference density state are included is in the thermal buoyancy term and the material is otherwise deemed incompressible) is obtained by setting $Di = 0$. Other slightly different forms of the equations are possible (see Tackley 1996).

A more general form for the Rayleigh number is given by

$$Ra = \frac{g_0 \alpha_0 F D^4}{\kappa_0^2 \nu_0} \quad \text{where} \quad \nu_0 = \frac{\eta_0}{\rho_0} \tag{23}$$

where $F$ is related to the heat flux that would be carried in the absence of heating: $|q| = \rho C_p F$. For internal heating only

$$F = \frac{|q|}{\rho C_p} = \frac{\rho h D}{\rho C_p} = \frac{hD}{C_p}$$

and $Ra$ is as above. For bottom heating which induces a difference in "potential" temperature across the system of $\Delta T$ then $F = \kappa \Delta T / D$ and

$$Ra = \frac{\alpha_0 g_0 \Delta T D^3}{\eta_0 \kappa_0}$$

(the potential temperature measures the temperature difference available to drive convection and does not include any adiabatic temperature rise). The Rayleigh number measures the strength of buoyancy forces relative to the strength of dissipative forces. The critical Rayleigh number that needs to be exceeded to get convection is on the order of 1000 (see Schubert et al, 2001, Chapter 7). For whole mantle convection $Ra$ is on the order of $10^9$. In this case, mantle convection is vigorous and turbulent. High viscosities at the base of the mantle may mean that the effctive Rayleigh number is depth dependent and that the critical Rayleigh number near the base of the mante is barely exceeded. Flow in such a region may be large-scale and laminar.

## 8. Summary of numerical convection experiments with variable viscosity

This is my summary of Tackley, JGR, 1996, v101, p3311

- Must do calculations in "wide" domain
- Get wide range of convective styles depending on $\eta(T, z)$, $k(z)$, $\alpha(z)$, etc.
- Boussinesq/basal: $\eta(T)$ leads to wide cells but upwelling sheets and downwelling plumes; $\eta(z)$ leads to small cells but now have upwelling plumes and downwelling sheets. As $Ra$ increases, we get smaller features and more time dependent flow.
- Compressible ($k(z)$, $\alpha(z)$,etc) /basal: large cells with large upwelling plumes, narrow downwelling sheets (except for $\eta(T)$ which is like Boussinesq).
- Compressible/internal: for $\eta$ constant we get short-wavelength flow with time dependent cold plumes *but* $\eta(T, z)$ leads to long wavelength flow with upwelling plumes and downwelling sheets.
- Temperature in the convecting cell is affected by rheology. The internal temperature is raised for $\eta(T)$ and $k(z)$ but is decreased for $\eta(z)$. The interior profile is roughly adiabatic
- Vertical variations in viscosity dominate planform
- Viscous dissipation and adiabatic heating are minor effects except for $\eta(T)$ when viscous dissipation can be important in the stiff upper boundary layer.
- Realistic plates do not develop and we get symmetric downwellings (need weak zones?)
- Effect of spherical geometry is minor except for obvious surface area effects (lower $T$ gradients are needed in the upper thermal boundary layer than in the bottom boundary layer because of the difference in surface area
- Power-law rheologies don't do much though there is a tendency to concentrate flow (could be important for rifting of the lithosphere due to plume impingement).
- Increas in viscosity at 660km can lead to long-wavelength flow (effective reduction of $Ra$).
- Phase transformations lead to layering and long-wavelength flow at high $Ra$ (get stronger layering and bigger avalanche effects)
- NEED PLATES