Elasticity and Flexure

1. Introduction

Elastic materials deform when a force is applied and they return to their original state (shape) when the force is removed. In the simplest cases, the *strain (deformation) is linearly proportional to the applied stress* **and** the material has elastic properties that are independent of direction (*isotropic*).

At stress levels that are high compared with the rock strength, deviations from elastic behavior are observed and the failure occurs. This can be either *brittle* failure or *plastic* failure. The mode of failure depends on whether the strength of the rock is governed by the confining pressure or by temperature, and whether the high stress levels are imposed over long or short time frames. Brittle failure results in faulting. Plastic or ductile failure or flow is an irreversible, continuous deformation without fracture. One example is long-term loading and unloading of Earth's lithosphere by ice caps – the fluid nature of the mantle causes rebound of topography following deglaciation (*e.g.*, Hudson Bay). Over long time scales, lower crustal rocks may deform plastically and so steep slopes *e.g.*at the edge of mountain ranges can "flow away". We shall see later that the properties of crustal and mantle rocks can result in different regions of brittle and ductile failure as a function of depth.

For terrestrial planets in general, rock strength depends on temperature, pressure, and composition. At depths of a few hundred kilometers and less, temperature and composition (especially volatiles) are the main effects. Thus the terrestrial planets can be expected to have a cold, strong (rigid) outer layer, or lithosphere, below which is mantle material that is weaker because of the strong decrease in strength with increasing temperature. Differences in lithospheric strength among the terrestrial planets reflect their different thermal environments and differing volatile contents of crustal and mantle rocks. A fundamental postulate of plate tectonics is that the Earth's lithosphere has sufficient strength to transmit stresses elastically over large horizontal distances, and so most terrestrial deformation occurs at plate boundaries (*i.e.*, is concentrated at weak zones). This is true to first order.

1:1. Investigating the long-term elastic behavior of the lithosphere

When subjected to loading (either surface loading through excess topography – *e.g.*, volcanoes; bending moments applied to the edges of plates; or loading from below the surface), a lithosphere with elastic strength will *bend*. The characteristics of the bending (flexural) response to loading provide information on the elastic properties of the lithosphere. Specifically, if we know something about the composition and rheology of the materials comprising the lithosphere we can use flexural reponses to loading to investigate the *lithospheric thickness*. Studies of the flexural response of the lithosphere to applied stresses also provide information on the stress state in the lithosphere. (See sketch of characteristic flexural response to loading: moat, outer rise)

1:2. Planetary examples of flexure - see figures in class

Note that the flexural response of the lithosphere may be observed not only in the topography but also in the gravity field – see later.

- Earth: Hawaiian islands, island chains and seamounts

- Earth: subduction zones

• Earth: folding in mountain belts

• Mars and Venus: loading by volcanoes

• Venus: loading by coronae

• Earth, Venus, Mars (and possibly Europa): rifting

Applications of flexural studies

• estimating lithospheric thickness, h_e

• estimating stresses in the lithosphere: σ_x, σ_y, σ_z

• from h_e we can get thermal gradient $\frac{dT}{dz}$

• from $\frac{dT}{dz}$ we get surface heat flow q_s

• variations in h_e temporally and spatially over a planet

1:3. Limitations of flexural modeling

(1) A purely elastic flexure model also assumes that the lithosphere can sustain infinite stresses; however, laboratory studies indicate that the strength of the upper lithosphere is limited by pressure-dependent brittle failure and the strength of the lower lithosphere is limited by temperature and strain-rate dependent ductile flow. Thus estimates from an elastic plate model of the *effective elastic thickness (EET)* of the lithosphere need to be corrected for more realistic rheological behavior.

(2) While the lithosphere may behave elastically over short to intermediate time scales, over longer timescales the lower viscosity of the mantle compared with that of the lithosphere may cause topographic signatures associated with flexure to relax. The resulting topography may still look like flexure but is in fact the combination of an initial elastic response of the lithosphere followed by viscous relaxation.

2. Linear Elasticity

A linear, isotropic solid is one in which stresses are linearly proportional to strains and mechanical properties have no preferred orientation. Principal axes of stress and strain coincide in such a material.

The generalized form of Hooke's Law in this case is

$$\sigma_{ij} = \lambda\epsilon_{ii}\delta_{ij} + 2\mu\epsilon_{ij} \tag{1}$$

σ_{ij} is the stress applied in the j'th direction to a plane with normal in the i'th direction. ϵ denotes strain, δ_{ij} is the Kroenecker Delta function such that $\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$ otherwise. λ and μ are collectively called the *Lame* constants. μ is also called the shear modulus or the modulus of rigidity (note that G is used for μ in Turcotte & Schubert's text).

In our system then we can write:

$$\sigma_{11} = \sigma_1 = (\lambda + 2\mu)\,\epsilon_1 + \lambda\epsilon_2 + \lambda\epsilon_3 \tag{2}$$

$$\sigma_{22} = \sigma_2 = (\lambda\epsilon_1 + (\lambda + 2\mu)\,\epsilon_2 + \lambda\epsilon_3 \tag{3}$$

$$\sigma_{33} = \sigma_3 = (\lambda\epsilon_1 + \lambda\epsilon_2 + (\lambda + 2\mu)\,\epsilon_3 \tag{4}$$

In engineering and geology we often use E, Young's modulus and ν, Poisson's ratio. These are related to λ and μ as follows

$$E = \frac{\mu\,(3\lambda + 2\mu)}{(\lambda + \mu)} \tag{5}$$

$$\nu = \frac{\lambda}{2\,(\lambda + \mu)} \tag{6}$$

and so equations 2 – 4 become

$$\epsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{7}$$

$$\epsilon_2 = \frac{-\nu}{E}\sigma_1 + \frac{1}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 \tag{8}$$

$$\epsilon_3 = \frac{-\nu}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{1}{E}\sigma_3 \tag{9}$$

A principal stress component σ produces a strain $\frac{\sigma}{E}$ in the same direction and strains $\frac{-\nu\sigma}{E}$ in the mutually orthogonal directions.

The elastic behavivor of a linear isotropic material can thus be specificed via E (elastic modulus reflecting the relative change in length of a material) and ν (the ratio of lateral strain to longitudinal strain in a body stressed longitudinally within its elastic limit). For geological materials Young's modulus varies from about 10 to 100 GPa and Poisson's ratio varies between about 0.1 and 0.4. Poisson's ratio for a fluid is 0.5 (see T&S Appendix 2, Section E for typical values of E, μ, and ν for various rocks).

3. Various special cases

Uniaxial stress – only one of the principal stresses, say σ_1 is non-zero. Typically many laboratory experiments on rocks are unaxial stress experiments.

Uniaxial strain – only one non-zero component of principal strain, say ϵ_1. Changes in stress due to sedimentation and erosion may use this approximation – the horizontal components of strain are assumed to be negligible c.f. the vertical component.

Plane stress – only one zero component of principal stress. Applications include studies of thermal stresses (due to temperature changes) in the lithosphere, where the region under stress is confined.

Plane strain – only one zero component of principal strain.

Isotropic Stress – all principal stresses are equal. Principal strains are also equal. Use to define bulk modulus and its reciprocal, the compressibility (see T&S, page 112). Application: determination of density with depth in the Earth.

4. Bending or flexure of plates

Review – lithosphere: strong outer layer; mantle beneath lithosphere is significantly weaker (lower viscosity) due to higher temperature and possibly the presence of volatiles. On Earth this region of the mantle is sometimes referred to as the asthenosphere.

Example of plate bending

Plate, thickness h, width L, infinitely long in z-direction pinned at ends and subjected to a line force V_a (N m^{-1}) at its center (Fig 3-9 from T&S goes below).

Static force balance and symmetry require equal vertical line forces $V_a/2$ applied at supports. Assume $h << L$ and deflection $w << L$ to be able to use linear elastic theory.

Outline of Derivation of Flexure Equation for 2-D Cartesian Geometry

We want to investigate how the deflection of the plate can be determined given the plate loading. Details are provided in T&S, pages 112 – 116. **You should read and understand this derivation.** Essentially one takes an element of the plate and balances the (1) applied forces ($q(x)$) with the net shear force ($V(x)$); and (2) the net bending moment ($M(x)$) with the torques due to horizontal forces ($P(x)$) and the shear force. From the moment and force balances one obtains:

$$\frac{d^2M}{dx^2} = -q + P\frac{d^2w}{dx^2} \tag{10}$$

This equation can be converted into a differential equation for the deflection, $w(x)$ if the bending moment M, can be related to w. It turns out that M is inversely proportional to the *local radius of curvature of the plate, R* and that R^{-1} is equal to $-\frac{d^2w}{dx^2}$ (see T&S for details).

......some intermediate steps lead to the (idealized, because elasticity assumed) relationship.....

$$M = \frac{-Eh^3}{12\left(1-\nu^2\right)}\frac{d^2w}{dx^2} \tag{11}$$

where E is Young's modulus, ν is Poisson's ratio, h is the elastic thickness of the plate. We often refer to the *flexural rigidity* of the plate, D, where

$$D = \frac{-Eh^3}{12\left(1-\nu^2\right)} \tag{12}$$

Thus the bending moment is the flexural rigidity of the plate divided by its curvature:

$$M = -D\frac{d^2w}{dx^2} = \frac{D}{R} \tag{13}$$

Substituting equation 15.12 into 15.10 we obtain a general equation for the deflection of an elastic plate:

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2} \tag{14}$$

Note that we have derived the flexure equation assuming a 2-dimensional (x,z) cartesian geometry. You should be aware that more general formulations for deflection of a spherical shell due to an imposed load are often required in planetary problems, in particular for small planets.

5. Application to planetary lithospheres

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2} \tag{15}$$

If you push down with an applied load $q_a(x)$ and displace material with density ρ_{below} with material ρ_{above}, the displacement will result in a restoring force (assuming $\rho_{below} > \rho_{above}$)

$$(\rho_{below} - \rho_{above})gw(x) = \Delta\rho g w(x)$$

The effective load is then

$$q(x) = q_a(x) - \Delta\rho g w(x)$$

So

$$D\frac{d^4w(x)}{dx^4} + P\frac{d^2w(x)}{dx^2} + \Delta\rho g w(x) = q_a(x) \tag{16}$$

1) For the oceanic case

$$\Delta\rho = \rho_{mantle} - \rho_{water}$$

2) For the continental case

$$\Delta\rho = \rho_{mantle} - \rho_{crust}$$

2) For loading by an isolated volcano

$$\Delta\rho = \rho_{mantle} - \rho_{air} \approx \rho_{mantle}$$

6. Elastic Plate Loading: General

The elastic response of the lithosphere to a load produces a characteristic flexural profile in topography. The load is surrounded by a depression or *moat*. Outboard of this depression is a *flexural bulge* or upwarp. We shall see that the wavelength of the flexural signature and the amplitude of the flexural bulge are related to the elastic plate thickness and the magnitude of the applied load.

One can clearly solve equation 15.16 for a range of different loading scenarios. Scenarios covered in Turcotte and Schubert include (1), (2) deflection of the lithosphere due to a load on a continuous or broken plate and (3) deflection of the lithosphere due to a bending moment applied to a plate. We will look at two situations here: loading of a continous plate by a simple line load (a good first order approximation to volcano loads), and loading of a plate by bending moments applied to a plate end (a good first order approximation to loading of terrestrial plates at subduction zones).

7. Bending of the lithosphere beneath volcanoes /seamounts

Figures 3-29 and 3-30 from Turcotte & Schubert:

We consider the behavior of the plate under a line load V_0 applied at $x = 0$ as shown above. $q_a(x) = 0$ everywhere except $x = 0$. The horizontal load $P = 0$. Equation 15.16 reduces to

$$D\frac{d^4w(x)}{dx^4} + \Delta\rho g w(x) = 0 \tag{17}$$

For oceanic islands $\Delta\rho = \rho_m - \rho_w$, for non-oceanic (more general planetary) cases $\Delta\rho = \rho_m$. The general solution of equation 15.25 is

$$w = e^{x/\alpha}\left(c_1 cos\frac{x}{\alpha} + c_2 sin\frac{x}{\alpha}\right) + e^{-x/\alpha}\left(c_3 cos\frac{x}{\alpha} + c_4 sin\frac{x}{\alpha}\right)$$

c_1, c_2, c_3, c_4 are determined by the boundary conditions and

$$\alpha = \left(\frac{4D}{\Delta\rho g}\right)^{(1/4)} \tag{18}$$

α is known as the flexural parameter.

Since there is symmetry about $x = 0$, we need only determine $w(x)$ for $x \geq 0$. $w(x) \rightarrow 0$ as $x \rightarrow \infty$, and $dw/dx = 0$ at $x = 0$. Thus c_1 and c_2 must be zero and $c_3 = c_4$. We get

$$w = c_3 e^{-x/\alpha}\left(cos\frac{x}{\alpha} + sin\frac{x}{\alpha}\right)$$

The constant c_3 is found by relating the line load to the flexural rigidity and the third derivative of the deflection (see T&S section 3-10, eqn 3-81 for details):

$$\frac{V_0}{2} = D\frac{d^3w}{dx^3}(x = 0) = \frac{4Dc_3}{\alpha^3}$$

So

$$w = w_0 e^{-x/\alpha}\left(cos\frac{x}{\alpha} + sin\frac{x}{\alpha}\right) \tag{19}$$

where

$$w_0 = \frac{V_0\alpha^3}{8D} \tag{20}$$

Now we look at the half-width of the flexural moat or depression, x_0. x_0 is the first "zero-crossing" of the deflection $w(x)$. Putting $w(x) = 0$ in equation 15.27 we can solve for x_0 in terms of α

$$x_0 = \alpha tan^{-1}(-1) = \frac{3\pi}{4}\alpha \tag{21}$$

Similarly the distance to the maximum amplitude of the flexural bulge x_b is obtained by setting $dw/dx = 0$ and solving for x_b in terms of α

$$\frac{dw}{dx} = -2\frac{w_0}{\alpha}e^{-x/\alpha}sin\frac{x}{\alpha} = 0$$

$$x_b = \pi\alpha \tag{22}$$

The height of the forebulge is obtained by substituting 15.30 into 15.27

$$w_b = -w_0 e^{-\pi} = -0.0432 w_0 \tag{23}$$

Note the dependence of α on h and of w_o on both V_0 and h.

8. Flexure of oceanic lithosphere at subduction zone

In this case one needs to include the load at one end ($x = 0$) of the plate, so ($q(x) = V_0$) and an applied horizontal bending moment per unit length, M_0. The solution to the general flexure equation

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^2w}{dx^2}$$

for this loading scenario is

$$w(x) = \frac{\alpha^2}{2D} e^{-\frac{x}{\alpha}} \left(-M_0 sin\left(\frac{x}{\alpha}\right) + (V_0\alpha + M_0)\, cos\left(\frac{x}{\alpha}\right)\right) \tag{24}$$

As for the case of loading of a continuous plate we can determine the distance (x_0) to the first "zero-crossing" of the topographic profile and the distance (x_b) to the maximum height of the outer rise or flexural bulge. We can also determine the maximum height (w_b) of the outer rise.

In practice we have a topographic profile that we wish to model to determine lithospheric thickness. The parameters M and V cannot be reliably estimated in practice, but we can estimate h_e by estimating x_0 and/or x_b. Alternatively in practice, a more common approach is to minimize the misfit between the predicted elastic plate deflections and topographic profile(s) across a given feature. This provides an estimate of the elastic plate thickness, and allows the computation of surface stresses, bending moments, and curvatures anywhere along a profile. Rember that the curvature, R, and the bending moment, M are given by

$$R = -\left(\frac{d^2w}{dx^2}\right)^{-1}$$

$$M = -D\frac{d^2w}{dx^2}$$

We can also calculate the horizontal surface stresses $\sigma_x x$ from

$$\sigma_{xx} = \frac{6M}{h_e^2} \tag{25}$$

where positive surface stresses correspond to tension and negative to compression.