Gravity - Additional Reading

1. Introduction

Force = mass x acceleration

Energy = force x distance

A body in a gravitational field is subject to gravitational forces (acceleration) AND has gravitational potential energy. We shall use both the gravitational potential and the gravitational acceleration as descriptions of a gravity field. They are equivalent, but sometimes one is more useful than another.

2. Inverse Square Law

The most useful thing to remember in the study of gravity is the *inverse-square law*, known as Newton's law of gravitation. Consider two point masses m_1 and m_2 separated by a distance r

The force on either mass, F, is

$$F = \frac{Gm_1m_2}{r^2} \tag{1}$$

G is called the "universal gravitational constant"

G = 6.673 $\pm$ .003 $\times$ 10^{-11} m^3 kg^{-1} s^{-2}

How do we use the inverse-square law? Suppose we have a complicated body. Consider a small element of the body of mass δm. The gravitational force δf exerted by the mass δm on an observer of mass m' at P is

$$\delta f = \frac{Gm'\delta m}{b^2} \tag{2}$$

The acceleration of the mass m' due to the element δm would be (remember force = mass x acceleration)

$$\delta g = \frac{\delta f}{m'} \tag{3}$$

so

$$\delta g = \frac{G\delta m}{b^2} \tag{4}$$

(remember that g is a vector so δg points from P towards δm. G is the gravitational constant.)

Equation 4 is the basic equation we use. For an arbitrary body g can be found by integrating over the mass distribution.

A body in a gravitational field has potential energy. The gravitational field is an example of a "conservative" force field. The differences in potential energy of a body between two points therefore depends only on the position of the points and not on the path along which the body has moved between them.

The *gravitational potential energy* of a body can be regarded as the negative of the work done on a body by a gravitational force of attraction in bringing it from infinity to its present position. The *gravitational potential* is the gravitational potential energy per unit mass.

$$E(r) = -\int_{\infty}^{r} F\,dx = \int_{r}^{\infty} F\,dx = -m\int_{r}^{\infty} g\,dx \tag{5}$$

so the gravitational potential, V, is

$$V = \frac{E(r)}{m} = -\int_{r}^{\infty} g\,dx$$

and

$$\frac{dV}{dr} = -g \tag{6}$$

SO: gravitational acceleration is the gradient of gravitational potential

Note that signs and notation can be confusing. Some people use dV/dr = g. Note that **g** is a vector. The bottom line is that only differences in potential are physically meaningful and you need to end up with **g** pointing in the right direction!

For those of you with more physics background you will see that in three dimensions we have the result **g** = ∇V.

3. Acceleration due to gravity for a spherically symmetric planet

Consider a spherical shell of radius r and thickness δr. Volume of the shell is $4\pi r^2\delta r$; mass, δm is $4\pi\rho(r)r^2\delta r$. The mass of a sphere of radius a is therefore

$$M = \int_{0}^{a} 4\pi\rho(r)r^2 dr \tag{7}$$

The gravitational force on a body of mass m at the surface of a spherically symmetric Earth is given by

$$F = \frac{GmM}{a^2} = mg \tag{8}$$

where M is the mass of the Earth and a is the radius of the Earth. g is the acceleration due to gravity of the body. Note the gravitational attraction is the same as that of a point mass, M.

Acceleration due to gravity at radius $r < a$ inside the Earth is

$$g(r) = \frac{G}{r^2} \int_0^r 4\pi\rho(x)x^2\,dx \tag{9}$$

Naturally, as we go away from the surface of a spherical body ($r > a$), the point mass assumption holds; the acceleration due to gravity outside the body is therefore

$$g(r) = \frac{GM}{r^2} \tag{10}$$

The equivalent gravitational potential is given by

$$V(r) = \frac{-GM}{r} \tag{11}$$

Note V is proportional to $\frac{1}{r}$ and g is proportional to $\frac{1}{r^2}$

DIGRESSION: Motivating Spherical Harmonic Descriptions of Global Fields

The previous description allowed us to find the gravitational *acceleration* on and gravitational *potential* of a body at a position, P, in a gravitational field. We considered a very simple example of a spherically symmetrical mass distribution. In general we will have a body with an arbitrary mass distribution – a planet with a spatially and radially varying density distribution – and we will want to describe the gravity field everywhere outside this arbitrary body. As mentioned before we will usually want to be able to go back and forth between descriptions of acceleration and of potential. To make such global descriptions we use what is called a *spherical harmonic expansion* that makes use of some basis functions called *Legendre Polynomials*. We need not worry about the details of these but let's motivate how they might arise as natural descriptions of a global gravity field.

Consider again the point mass example and the potential U at a point P, due to a mass m, at location Q:

$$V(r) = \frac{-Gm}{s} \tag{12}$$

where s is the distance from P to Q.

In many cases we may need to describe this potential relative to a reference frame with origin O.

From the law of cosines:

$$s^2 = r_1^2 + r_2^2 - 2r_1 r_2 cos(\gamma) \tag{13}$$

Divide through by the maximum of r_1 or r_2, here r_1

$$z^2 = 1 + x^2 - 2xt \tag{14}$$

where $z = \frac{s}{r_1}$, $x = \frac{r_2}{r_1}$, $t = cos(\gamma)$

One now does a Taylor series expansion on the function $\frac{1}{z}$.

Remember: $(1 + y)^n = 1 + ny + \frac{n(n-1)y^2}{2!} + ...$

$$(1 + x^2 - 2xt)^{-\frac{1}{2}} = 1 + tx + \frac{1}{2}(-1 + 3t^2)x^2 + \frac{1}{3}(......)x^3....... \tag{15}$$

The coeeficients of each degree in x are polynomials in t, *i.e.*in $cos(\gamma)$. These are the Legendre polynomials.

Thus we can write the gravitational potential of a point mass m, as an infinite series. One can now think of the generalized case of a body composed of various point masses $m_i, i = 1, ...N$; *i.e.*the body can have an arbitrary internal density distribution and arbitrary shape. We sum all the inifinite series for the potential due to each point mass m_i to get the total potential. So:

$$U = \frac{-G}{s} \sum_{i=1}^{N} m_i \left(1 + tx + \frac{1}{2}(-1 + 3t^2)x^2 + \frac{1}{3}(......)x^3.......\right) \tag{16}$$

Spherical Harmonic Descriptions of Gravity Fields - continued

Thus for a "real planet" we typically describe (or parameterize) the gravitational potential in 3-D space in terms of Legendre polynomials, P_l.

$$\begin{aligned} V(r,\theta) = -\frac{GM}{r}[J_0 P_0(\cos\theta) - \frac{a}{r} J_1 P_1(\cos\theta) \\ -\frac{a^2}{r^2} J_2 P_2(\cos\theta) - \frac{a^3}{r^3} J_3 P_3(\cos\theta)..] \end{aligned} \tag{17}$$

(a is the equatorial radius of the planet; θ is the colatitude). In this expansion, the J's are empirically determined coefficients and the P's are “Legendre polynomials.” The first few are

$$
\begin{aligned}
P_0(\cos\theta) &= 1 \\
P_1(\cos\theta) &= \cos\theta \\
P_2(\cos\theta) &= \frac{1}{2}(3\cos^2\theta - 1) \\
P_3(\cos\theta) &= \frac{\cos\theta}{2}(5\cos^2\theta - 3)
\end{aligned}
$$

These functions are progressively wigglier (as l, the spherical harmonic degree, increases) functions of the colatitude θ and we can choose the J's so that the sum can approximate very accurately any observed dependence of V on θ. *Even-l* terms are symmetric about the equator, *odd-l* terms are antisymmetric about the equator.

Aside: For those of you familiar with Laplace's equation you will notive that equation 17 is the solution to Laplace's equation in spherical coordinates, $\nabla^2 V = 0$. Laplace's equation describes the gravitational potential anywhere exterior to a planet (*i.e.*, in a source-free region).

4. Gravity and Potential Due to Ellipsoid: Shape Effect

Equations 10 and 11 are good first order approximations to the value of g and V on the Earth, but because the Earth is not exactly spherical, there are small variations. The biggest variation is due to the equatorial bulge of the Earth. This bulge is caused by the Earth's rotation. On long time scales the Earth behaves as a fluid and the spinning of the Earth causes it to have the shape of an oblate spheroid. Here we first examine the ellipticity, then we add in the correction due to rotation.

The effect of flattening

The "flattening" of the Earth, f, is defined by

$$
f = \frac{a - c}{a} = 3.353x10^{-3} \tag{18}
$$

Thus the distortion of the Earth from a sphere is very small, so the perturbation of g at the surface away from the value GM/a^2 will also be very small. g will be a weak function of latitude but not a function of longitude, because the rotationally distorted Earth is symmetric about the axis of rotation. A consequence of the distortion is that g no longer points directly towards the center of the body. There will be more mass at the equator than the poles so g will be deflected towards the bulge – it has a small tangential component as well as a radial component.

$$
\begin{aligned}
g_r &= \frac{GM}{r^2} + \delta g_r \\
\delta g_r &\ll \frac{GM}{r^2}
\end{aligned}
$$

We now have

$$
g = (g_r^2 + \delta g_t^2)^{\frac{1}{2}} = \left[(\frac{GM}{r^2} + \delta g_r)^2 + \delta g_t^2 \right]^{\frac{1}{2}}
$$

With some math (binomial expansions and first-order approximations in δg_r and δg_t - see Turcotte and Schubert, p 197–198 for details)

$$g = \frac{GM}{r^2}[1 + \frac{1}{2}\epsilon] = \frac{GM}{r^2}\left[1 + \frac{\delta g_r r^2}{GM}\right] = \frac{GM}{r^2} + \delta g_r = g_r \tag{19}$$

We can neglect the transverse component, δg_t, and only need consider δg_r.

δg_r is a function of latitude. The full solution for the latitude dependence requires the use of spherical harmonic analysis. We use our prior motivation for spherical harmonic descriptions to outline the approach.

From previously we saw the solution to Laplace's equation can be written as a series expansion:

$$V(r,\theta) = -\frac{GM}{r}[J_0 P_0(\cos\theta) - \frac{a}{r}J_1 P_1(\cos\theta) - \frac{a^2}{r^2}J_2 P_2(\cos\theta) - \frac{a^3}{r^3}J_3 P_3(\cos\theta)\cdots]$$

Consider the shape of $P_1(\cos\theta)$ as a function of colatitude. This function is antisymmetric about the equator – our equatorial bulge is symmetric about the equator. This term isn't going to help us fit the gravitational potential because it doesn't have the right geographical dependence – we can therefore set $J_1 = 0$. (If we choose the center of mass as the origin of our coordinate system then $J_1 = 0$ by definition.)

Now let us examine $P_2(\cos\theta)$. This function is symmetric about the equator and has roughly the shape of an equatorial bulge. We would therefore expect J_2 to be important. In fact, when we fit the observed latitude dependence of V we find that J_2 is over 500 times bigger than any of the other J's so to a very good approximation we have

$$V(r,\theta) = -\frac{GM}{r}\left(1 - \frac{a^2}{r^2}J_2 P_2(\cos\theta)\right)$$

$$= -\frac{GM}{r}\left(1 - \frac{a^2}{r^2}\frac{J_2}{2}(3\cos^2\theta - 1)\right)$$

$$V(r,\theta) = -\frac{GM}{r}\left(1 - \frac{a^2}{r^2}\frac{J_2}{2}(3\sin^2\lambda - 1)\right) \tag{20}$$

where λ is the latitude. ($\theta = \frac{\pi}{2} - \lambda$ so $\cos\theta = \sin\lambda$)

To first order we found that g still points radially inwards so we can calculate g (the acceleration due to gravity) by using the relationship $g = dV/dr$, *i.e.*

$$g(r,\theta) = \frac{GM}{r^2} - \frac{3GMa^2}{2r^4}J_2(3\sin^2\lambda - 1) \tag{21}$$

This equation gives g at a radius r above the center of the Earth and the values of the constants used are

$$GM = 3.986005x10^{14}\ \mathrm{m}^3\ \mathrm{s}^{-2}$$
$$a = 6378.139\mathrm{km}$$
$$J_2 = 1.08270x10^{-3}$$

(Note that J_2 is still much less than 1 even though it is much larger than any of the other J's.)

It turns out that J_2 is related to the *moments of inertia* of the planet and is helpful in constraining the density distribution. $g(r, \theta)$ can be measured accurately from tracking the orbits of artificial satellites, allowing the measurement of J_2.

5. Gravity and Potential Due to Ellipsoid: Effect of Centrifugal acceleration

Equations 19 and 20 cannot be used on the surface of the Earth because the Earth is rotating. An object on the Earth's surface is subject to centrifugal acceleration and so we need an additional correction for this.

The centrifugal acceleration, g_ω, is given by

$$g_\omega = \Omega^2 s \tag{22}$$

We want the component of g_ω in the radial direction. g_ω points outwards in a direction perpendicular to the rotation axis so the radial component is $g_\omega \cos\lambda$. From the diagram you can also see that $s = r\cos\lambda$ so the radial component outward of g_ω is given by

$$\Omega^2 r \cos^2\lambda \tag{23}$$

We have been using the sign convention that a positive value of g points radially inwards so the centrifugal acceleration acts to counteract this, *i.e.*on the surface of the Earth

$$g(r, \lambda) = \frac{GM}{r^2} - \frac{3GMa^2}{2r^4} J_2 (3\sin^2\lambda - 1) - \Omega^2 r \cos^2\lambda \tag{24}$$

We can also redefine the gravitational potential at the Earth's surface to include the centrifugal potential. This is sometimes called the geopotential and is given the symbol U instead of V. We get U by integrating the expression for g – the first two terms are the same as for V but now we have an extra term:

$$U(r, \lambda) = -\frac{GM}{r} + \frac{GMa^2 J_2}{2r^3}(3\sin^2\lambda - 1) - \frac{1}{2}\Omega^2 r^2 \cos^2\lambda \tag{25}$$

6. Measurement of planetary gravity fields and geoid

Earth

The long wavelength geoid can be found by the same procedures we discussed for J_2 for the Earth. Careful records are kept of artificial satellite orbits, whose positions in space are triangulated from the ground via their radio time-of-flight delays. Then the equations of motion for a small body in a gravitational field are solved, taking account of things like corrections to the orbit from perturbations due to the gravitational attractions of the sun and moon. The results provide global geoid and gravity maps. One way to think about the differences between the geoid and gravity signal is that to first order the graviational acceleration is proportional to $\frac{1}{r^2}$, while the geoid is a measure of potential proportional to $\frac{1}{r}$. Thus one can think of the geoid as being more sensitive to density anomalies at greater depth in the Earth: this is reflected in the enhancement of long wavelength anomalies relative to the gravity signal. More recently the satellites Geosat and Seasat have been equipped with radar altimeters capable of measuring the satellite altitude to an accuracy of a few centimeters above a mean ocean (the altimeters have a footprint several km^2 in area). These measurements provide short wavelength information (scales less than 500km), mostly about submarine topography, since the geoid is elevated over topographic heights, and can be used to find high resolution gravity anomaly.

Venus, Moon, Mars

For Venus, the Moon and Mars gravity data have been obtained from two-way radar Doppler tracking of spacecraft orbiting these planets (Magellan, Mars Global Surveyor, Clementine and Lunar Prospector) at Deep Space Network stations (Goldstone, California; Madrid, Spain; Canberra, Australia). Factors affecting the accuracy and resolution of the gravity field models obtained for these planets include the distance of the orbiting spacecraft above the planet (remember that $g(r)$ falls off as $\frac{1}{r^2}$ and so the orbital perturbations of the spacecraft due to density anomalies in the planet fall off quickly with spacecraft height. Noise in the Doppler signals arises from many sources including solar effects, instrument noise, atmospheric effects, etc.

The Doppler data are processed to obtain spherical harmonic gravity models. An orbit determination program is used, which calculates spacecraft acceleration, velocity and position due to gravitational forces of the planet (the spherical harmonic model) as well as time and space dependent drag forces (e.g., due to atmospheric density).

7. Moments of inertia

Motivation and Measurement

Planetary moments of inertia provide important constraints on the internal density distribution of the planetary body. Estimates of moments of inertia require two related observations: (1) amplitudes of the spherical harmonic degree two gravity field (see previous lecture for spherical harmonic analysis motivation and description – one of the degree two terms for the gravity field is J_2); (2) rotational responses to known torques, in particular precession of the spin axis. These quantities are best measured by spacecraft or artificial satellites orbiting a planet.

Spherically symmetrical planet

We consider the moment of inertia of a spherically symmetrical planet, *i.e.*density is just a function of radius. Spherical symmetry means that the moment of inertia about any axis going through the center of the planet will be the same. For simplicity, we choose the rotation axis to compute the moment of inertia.

Note: For a general non-spherically symmetric density distribution we calculate the three principal moments of inertia of the planet. These are the moments of inertia about the x, y, and z axes. These are called A, B and C respectively. For a spherically symmetric planet it is clear that $A = B = C$. Here we compute C.

It is useful to use a spherical polar coordinate system which is related to the cartesian coordinate system in the following way.

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta$$

An element of volume

$$\begin{aligned} dV &= dxdydz \quad \text{in a cartesian coordinate system} \\ &= r^2\sin\theta dr d\theta d\phi \quad \text{in a spherical coordinate system} \end{aligned}$$

Now reconsider the planet.

The mass element centered at point, P is a perpendicular distance s from the z axis. It has a moment of inertia about the z axis given by $s^2 dm$. The total moment of inertia of the planet about the z axis is the integral of all such elements over the entire mass distribution. So

$$C = \int_M s^2\, dm$$

We have

$$dm = \rho(r)\, dV$$

and also

$$\begin{aligned} s^2 = x^2 + y^2 &= r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) \\ &= r^2\sin^2\theta \end{aligned}$$

So now we can write

$$C = \int_V r^2\sin^2\theta\rho(r)\, dV$$

where the integral is over the volume of the planet. This integral can be written as one over r (from 0 to a), over θ (from 0 to π) and over ϕ (from 0 to 2π) by substituting in the expression for dV:

$$C = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho(r) r^4 \sin^3\theta\, dr d\theta d\phi$$

We can do these integrals separately. You will note that nothing in the integrand ($\rho r^4 \sin^3\theta$) depends upon ϕ, the longitude, so we can write

$$C = \int_0^\pi \int_0^a \rho(r) r^4 \sin^3\theta \, dr d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^\pi \int_0^a \rho(r) r^4 \sin^3\theta \, dr d\theta$$

This can also be written as

$$C = 2\pi \int_0^a \rho(r) r^4 \, dr \int_0^\pi \sin^3\theta \, d\theta$$

The last integral is simple to do if we let $f = \cos\theta$; we can write $df/d\theta = -\sin\theta$ and f varies from 1 to -1 as θ varies from 0 to π, *i.e.*$df = -\sin\theta d\theta$ so

$$\int_0^\pi \sin^3\theta \, d\theta = -\int_1^{-1} \sin^2\theta \, df = -\int_1^{-1} (1 - \cos^2\theta) \, df$$

$$= -\int_1^{-1} (1 - f^2) \, df$$

$$= -\left[f - \frac{f^3}{3} \right]_1^{-1} = \frac{4}{3}$$

Finally, we have

$$C = \frac{8\pi}{3} \int_0^a \rho(r) r^4 \, dr \tag{26}$$

Thus the moment of inertia tells us something about the density distribution within the planet. Note, however, that unless we have other a priori information about the planet (such as the mean density, the planet's size, etc) there are many different density distributions that can give us the measured value of C. This is because C is proportional to the integral of the product $\rho(r)$ and r^4.

Suppose that a planet has a *uniform density*, $\overline{\rho}$, then

$$C = \frac{8\pi}{3}\overline{\rho}\int_0^a r^4\,dr = \frac{8\pi}{15}\overline{\rho}a^5 \tag{27}$$

The datum that is used is usually not C but C/Ma^2 which is a dimensionless number (M is the mass of the planet). Now

$$M = \frac{4}{3}\pi a^3\overline{\rho}$$

So for a uniform density planet

$$\frac{C}{Ma^2} = \frac{8\pi}{15}\overline{\rho}a^5\frac{3}{4\pi a^3\overline{\rho}}\frac{1}{a^2} = \frac{2}{5} = 0.4$$

If the density is greater near the center of the planet we find that $C/Ma^2 < 0.4$. For example C/Ma^2 for the Earth is .3308. See attached table for other known mean moments of inertia.

Principal moments of inertia for general planetary bodies

On a rotationally distorted Earth we have $A = B$ (because of the symmetry about the rotation axis) but now $A \neq C$. For a more general planet $A \neq B \neq C$. Mars is a good example of such a planet – it is ellipsoidal due to rotation, but the mass distribution is not even symmetrical about the rotation axis due to the large excess mass associated with the Tharsis rise.

How might we measure C? A straightforward, but tedious, calculation allows J_2 to be cast in terms of the principal moments of inertia of the body (the details are on page 199–200 of Turcotte and Schubert - note how to calculate "A"). We find that

$$J_2 = \frac{C - A}{Ma^2} \tag{28}$$

This equation allows us to estimate $C - A$ because J_2 is measured. For the Earth the gravitational attractions of the Sun and the Moon, acting on the equatorial bulge, cause a precession of the axis of rotation. From the rate of precession we can find the “dynamical ellipticity”, H, which is given by

$$H = \frac{C - A}{C} \tag{29}$$

H is estimated to be 1/305.51 and by combining equations we have

$$C = \frac{J_2}{H}Ma^2 = .3308Ma^2 \tag{30}$$

8. The geoid

An "equipotential surface" is a surface on which U is a constant. The vector **g** will be normal to any such surface, thus U defines the local horizontal. For Earth, the sea surface is an equipotential surface (apart from the effects of wind and currents), and the equipotential surface which defines sea level is called the *geoid*.

We can define reference states for the gravity field and the potential based on the approximate shape and rotation of a planet (see lecture 6). One can go through some Math (for those of you who have taken ES 103 see the notes and associated homework problem; also see Turcotte and Schubert pp 202-203) to calculate the actual surface, r_0 that corresponds to the potential for a rotating ellipsoid that we calculated in the last lecture:

$$U_0(r_0, \lambda) = -\frac{GM}{r_0} + \frac{GMa^2 J_2}{2r_0^3}(3\sin^2\lambda - 1) - \frac{1}{2}\Omega^2 r_0^2 \cos^2\lambda$$

$$r_0 = a\left[1 + \frac{(2f - f^2)}{(1-f)^2}\sin^2\lambda\right]^{-\frac{1}{2}} \tag{31}$$

This is the reference geoid. f is the flattening and is given by $f = \frac{a-c}{a}$ or alternatively by $f = \frac{3J_2}{2} + \frac{a^3\Omega^2}{2GM}$. Note that for Earth r_0 is almost a spherical surface ($f << 1$; for Earth $F \approx 1/300$). Observed departures of the actual geoid from the reference geoid are called "geoid anomalies."

If we plot the actual geoid (for Earth, the equipotential surface which most closely coincides with the sea surface) and the reference geoid they may differ. The difference between these, ΔN, is a *geoid anomaly* (also called the *geoid height*) and is measured in meters.

Geoid height is directly related to local geopotential as follows: local gravity is given by

$$\mathbf{g} = -\hat{\mathbf{r}}\frac{\partial \mathbf{U}}{\partial \mathbf{r}}.$$

Clearly we have $|\mathbf{g}| = \mathbf{g} = \frac{\partial \mathbf{U}}{\partial \mathbf{r}}$. Remember by our definition U is negative, and increases as we move away from Earth. Suppose we call the reference geoid U_R then the observed geoid U_0 can be computed in terms of the properties of the reference geoid using a Taylor series expansion:

$$U_0 = U_R + \left(\frac{\partial U_R}{\partial r}\right)_{r_0} \Delta N$$

Now the radial derivative of the potential is the acceleration due to gravity so we can define a reference gravity field, g_R, (the acceleration due to gravity on the reference geoid) such that

$$g_R = \left(\frac{\partial U_R}{\partial r}\right)_{r_0}$$

An anomaly in the potential ΔU is defined as

$$\Delta U = U_R - U_0 = -g_R \Delta N,$$

and the geoid anomaly ΔN is

$$\Delta N = -\frac{\Delta U}{g_R}$$

Since g varies by less than a percent the above is an excellent approximation, that is geoid height is simply proportional to minus local geopotential anomaly. *Note that a local mass excess produces an outwarp of gravity equipotentials and therefore a positive ΔN and a negative ΔU.*

9. The effect of topography on gravity – Bouguer anomaly

Mass anomalies associated with topograph give rise to surface gravity anomalies. The effect of general topography must be treated numerically (sometimes called a terrain correction) – integration of $\delta g_m = \frac{G\delta m}{b^2}$ – but, if the topography is slowly varying (*i.e.*it has a shallow slope) we can derive an approximate expression for the gravitational effect due to topography.

We consider a cylindrical disc of material of radius R and thickness h. An observer is located a distance b above the upper surface of the disc. The density in the disc is assumed to be just a function of depth, *i.e.*$\rho = \rho(z)$

Because of the symmetry of the disc we know that the net gravitational attraction will be vertically downwards (g_z). We consider the contribution δg_z to g_z due to a cylindrical ring of radius r and thickness dr inside the disc at a depth z. The volume of the ring is $2\pi r dr dz$ so the mass of the ring is $2\pi r dr dz \rho(z)$. The observer is a distance $b + z$ from the center of the ring and so is a distance of $(r^2 + (z + b)^2)^{1/2}$ from a segment of the ring. To get the mathematical expression for δg_z we divide the ring into segments.

The volume of a segment is $r d\phi dr dz$. The volume of the whole ring is therefore $\int_0^{2\pi} r dr dz d\phi = 2\pi r dr dz$. The gravitational attraction of the segment at the observer who is a distance $(r^2 + (z + b)^2)^{1/2}$ away (see Figure 4.19) is

$$\frac{G}{r^2 + (z+b)^2}\delta m = \frac{G}{r^2 + (z+b)^2} r \, d\phi dr dz \rho(z)$$

This points directly towards the segment from the observer. The contribution to the vertical component of g (*i.e.*δg_z) is found by multiplying by $\cos\theta$ but

$$\cos\theta = \frac{z+b}{[r^2 + (z+b)^2]^{1/2}}$$

The total contribution of the ring, δg_z, is found by summing up all the contributions of the segments in the ring which is equivalent to integration over ϕ from 0 to 2π.

Putting all these bits together gives

$$\delta g_z = \int_0^{2\pi} \frac{G}{r^2 + (z+b)^2} r dr dz \rho(z) \cos\theta d\phi$$
$$= \frac{2\pi G r \rho(z) dr dz}{r^2 + (z+b)^2} \frac{z+b}{[r^2 + (z+b)^2]^{1/2}}$$

(Note that the angle θ is the same for all segments of the ring so θ doesn't depend upon ϕ.)

To get the total gravitational attraction of the disc we sum up the contributions of all the rings – this is accomplished by integrating the expression for δg_z over r (from $r = 0$ to R) and over z (from $z = 0$ to h), *i.e.*

$$g_z = 2\pi G \int_0^h \int_0^R \frac{\rho(z) r (z+b)}{[r^2 + (z+b)^2]^{3/2}} dr dz$$
$$= 2\pi G \int_0^h \rho(z)(z+b) \left\{ \int_0^R \frac{r}{[r^2 + (z+b)^2]^{3/2}} dr \right\} dz$$

Integrating with respect to r gives

$$g_z = 2\pi G \int_0^h \rho(z) \left[1 - \frac{z+b}{[R^2 + (z+b)^2]^{1/2}} \right] dz$$

In the limit that the disc is very broad (*i.e.*$R \to \infty$) we have a slab of topography of thickness h. This is a good approximation if the topography is shallow sloped. In this case,

$$g_z \simeq 2\pi G \int_0^h \rho(z)\, dz \tag{32}$$

This is called the *Bouguer gravity formula*. If topography has a height h and a (constant) density ρ_c its contribution to g is

$$\Delta g = 2\pi G \rho_c h \tag{33}$$

This corrects gravity measurements for the effect of mass excess or deficit due to topography.

Additionally, if we make a gravity measurement at various elevations, we are at different distances from the center of mass of the Earth and so g changes. We can correct for this effect.

Suppose we make a gravity measurement at a particular position on the surface of the Earth. To calculate the gravity anomaly we would first subtract out the reference gravity field, g_R, (which has a latitude dependence) evaluated on the reference geoid which is at r_0. We may calculate the effect of elevation in the following way. To zeroth order we have

$$g = \frac{GM}{r^2}$$

so the effect of changing distance from the center of a spherical earth is:

$$\frac{\delta g}{\delta r} = -2\frac{GM}{r^3} = 2\frac{g}{r}$$

therefore the anomaly caused by elevation is approximated by:

$$\Delta g_h = \frac{2h}{r_0} g_R$$

*i.e.*gravity is reduced by an amount Δg_h where

$$\Delta g_h = \frac{2hg_R}{r_0} \tag{34}$$

If we add Δg_h to our measurement we have corrected for the effect of elevation. This correction is called the *free-air correction* and a gravity anomaly defined by

$$\Delta g_{fa} = g_{obs} - g_R + \Delta g_h \tag{35}$$

is called a *free-air gravity anomaly*. If we make a further correction for the effect of topography using the Bouguer gravity formula we have a new gravity anomaly defined by

$$\Delta g_B = \Delta g_{fa} - 2\pi G \rho_c h \tag{36}$$

This is called a *Bouguer gravity anomaly*.

10. Compensation and isostatic geoid anomalies

The Bouguer gravity formula removes the effect of topography as if the topography were uncompensated. Short wavelength features can be supported by the lithosphere and so the Bouguer correction effectively accounts for the gravitational attraction of the anomalous mass. Long wavelength topography (*e.g.*a mountain range) causes the lithosphere to “sag" into the mantle. Since the Moho (crust /mantle boundary is generally embedded in the lithospehre) such “compensated" features have low density roots associated with them. This results in a large negative Bouguer anomaly:

Let us suppose that topography is perfectly *isostatically compensated*. If the mantle behaves like a fluid, the pressure will be constant along level surfaces at depth. If the pressure were not constant the mantle material would flow until the lateral pressure variation was removed. In this case we know that, at some depth, the total mass in vertical columns of material must be equal. This depth is called the *depth of compensation*. In our picture the total mass in a column of material under A is the same as the total mass in a column under B. This means that the free-air anomaly must be zero *i.e.*the only variation in g is due to the fact that at position B we are further from the center of the Earth than at position A. The condition for isostatic compensation can be written as

$$\int_0^h \Delta\rho(z)dz = 0 \tag{37}$$

where $\Delta\rho$ is the difference in density between two columns as a function of depth and h is the depth of compensation. This equation tells us something about lateral variation in density as a function of depth. It turns out that the *geoid* anomalies associated with isostatic compensation give us more information about $\Delta\rho(z)$.

To compute the geoid anomaly over topography we proceed as we did in computing the Bouger gravity anomaly. The increment in U caused by a ring of material is

$$\delta U = -\frac{2\pi G r \Delta\rho(z) dr dz}{[r^2 + (z+b)^2]^{1/2}} \qquad \left[\text{from} \quad \delta U = -\frac{G\delta m}{x}\right]$$

The anomaly in the potential caused by a disc of material is found by integrating this expression over z ($0 \rightarrow h$) and over r ($0 \rightarrow R$). After some intermediate steps (you need not worry about the details of the derivation) and assuming again (as in the Bouger correction calculation) that the disk is broad compared with the observation height we get

$$\Delta U = -2\pi G \left\{ R \int_0^h \Delta\rho(z)dz - \int_0^h \Delta\rho(z)(z+b)dz \right\}$$

$$= -2\pi G \left\{ R \int_0^h \Delta\rho(z)dz - \int_0^h \Delta\rho(z)z dz - b \int_0^h \Delta\rho(z)dz \right\}$$

For perfect isostatic compensation, we have

$$\int_0^h \Delta\rho(z)dz = 0$$

so

$$\Delta U = 2\pi G \int_0^h z\Delta\rho(z)dz \tag{38}$$

The gravitational potential anomaly due to a shallow long wavelength isostatic density distribution is proportional to the dipole moment of the density distribution below the measurement point.

Geoid anomalies (ΔN) are measured in meters and are related to anomalies in potential by using the equation developed earlier:

$$\Delta N = -\frac{\Delta U}{g_R} = -\frac{2\pi G}{g_R} \int_0^h z \Delta\rho(z) dz \tag{39}$$

where g_R is gravity on the reference geoid. We can use this equation to tell us about $\Delta\rho(z)$ – in fact it can be used to distinguish between different models of compensation.

11. Airy compensation

Isostatic compensation can be achieved in several ways. We discuss three simplified models but in reality compensation is probably achieved by a complicated mixture of these models (and of other effects).

In Airy isostatic compensation is achieved by varying the thickness of a constant density crust. The thickness of continental crust with zero elevation (*i.e.*the surface at sea level) is H. Topography of height h is associated with a low density root of thickness b. Using isostasy (principle of hydrostatic equilibrium) we have

$$\rho_c H + \rho_m b = \rho_c(h + H + b)$$

which we can rearrange to give

$$b = \frac{\rho_c h}{\rho_m - \rho_c} \tag{40}$$

The geoid anomaly associated with compensated positive topography can also be found. If we choose sea level (or whatever our reference surface is) as our origin and note that $\Delta\rho(z)$ is $\rho(z)$ in column A minus $\rho(z)$ in column B we have

$$\begin{aligned}
\Delta N &= -\frac{2\pi G}{g_R}\left[\int_{-h}^{0} \rho_c z dz + \int_0^H (\rho_c - \rho_c) z dz + \int_H^{H+b} (\rho_c - \rho_m) z dz\right] \\
&= -\frac{2\pi G}{g_r}\left[-\frac{\rho_c h^2}{2} + \left(\frac{\rho_c - \rho_m}{2}\right)[(H+b)^2 - H^2]\right] \\
&= +\frac{\pi G}{g_r}\left[\rho_c h^2 + (\rho_m - \rho_c)(2Hb + b^2)\right]
\end{aligned}$$

we can now eliminate b using equation 7.15 giving

$$\Delta N = \frac{\pi G}{g_R}\rho_c\left[2Hh + \frac{\rho_m h^2}{\rho_m - \rho_c}\right] \tag{41}$$

If $\frac{h}{H} << 1$, then the second term can be neglected and we can estimate the geoid-to-topography ratio (GTR)

$$GTR = \frac{\Delta N}{h} = \frac{2\pi G}{g_0}\rho_c H \tag{42}$$

We see there is a simple linear relationship between GTR and the depth of compensation, H.

This kind of model (crustal thickening) can explain geoid anomalies for several features of the terrestrial planets:

(1) Earth: passive continental margins - see T&S, figure 5-21

(2) Earth: several major mountain belts, *e.g.*the Himalayas

(3) Mars: the elevation of the southern highlands

(4) Venus: flat plateau-like regions, *e.g.*, Ovda Regio

(5) the Moon: the lunar highlands

12. Pratt compensation

An alternative way to achieve isostatic compensation is to have horizontal variations in density over some prescribed depth range, W. (*i.e.*, the denisities of adjacent blocks are different, but the block bases are all at the depth of compensation). Suppose the reference density corresponding to no topography is ρ_0 and suppose the p^{th} column has positive topography of height h and density ρ_p. Then

$$\rho_0 W = \rho_p(W + h)$$

which we can rearrange to give ρ_p

$$\rho_p = \frac{\rho_0 W}{W + h} \tag{43}$$

The geoid anomaly is

$$\Delta N = -\frac{2\pi G}{g_R}\int_0^h z\Delta\rho(z)dz$$

Here $\Delta\rho$ is the density difference betweeen the p^{th} column and the reference density. Thus

$$
\begin{aligned}
\Delta N &= -\frac{2\pi G}{g_R}\left\{\int_{-h}^{0}\rho_p z dz + \int_{0}^{W}(\rho_p - \rho_0) z dz\right\} \\
&= -\frac{2\pi G}{g_R}\left[\frac{-\rho_p h^2}{2} + (\rho_p - \rho_0)\frac{W^2}{2}\right]
\end{aligned}
$$

Eliminating ρ_p gives

$$
\Delta N = \frac{\pi G}{g_R}\rho_0 W h \tag{44}
$$

In this case ΔN is proportional to h.

13. Thermal Isostacy

Long-wavelength topography may be supported by lateral density variations within the mantle rather than or as well as by lateral density variations in the crust. To support high topography buoyant forces in the mantle may arise from either compositional or temperature variations (so high topography is supported by either mantle that is regionally less dense by virtue of either being compositionally distinct or hotter).

Earlier we saw

$$
\Delta N = -\frac{2\pi G}{g_R}\int_{0}^{h} z\Delta\rho(z) dz
$$

In the case of thermal isostacy $\Delta\rho(z) = -\rho_0\alpha\Delta T(z)$, where α is the coefficient of thermal expansion (assumed to be constant over the dpeth range of the mantle considered), ρ_0 is the ambient density, and ΔT is the temperature contrast with the surrounding mantle (so $\Delta T(z) = T(z) - T_m$). The negative sign arises since an increase in temperature results in a decease in density. We assume that the density anomaly persists from the base of the crust H, downwards to some depth $H + m$. Thus

$$
\begin{aligned}
\Delta N &= -\frac{2\pi G}{g_R}\left\{\int_{-h}^{0}\rho_c z\, dz + \int_{H}^{H+m} -\rho_m\alpha(T(z) - T_m) z\, dz\right\} \\
&= -\frac{2\pi G}{g_R}\left\{\frac{\rho_c h^2}{2} - \rho_m\alpha\int_{H}^{H+m}(T(z) - T_m)\, z\, dz\right\}
\end{aligned}
$$

Thus if we know $T(z)$ we can compute the integral above.