

Scattering from weak heterogeneity—the Born approximation

We have just seen how the idea of secondary sources as developed in Kirchhoff theory provides a way to obtain solutions for waves interacting with a rough interface. In our Kirchhoff formulas there is no limit regarding the size of the velocity contrast that may be present across the interface; the Kirchhoff approximation is accurate provided the interface is not so rough that multiple scattering becomes important. In the case of weak heterogeneity, there is another equivalent source theory that can be applied. The theory is based on the *Born approximation* for single scattering in weakly heterogeneous media. In this method, we assume that the wavefield consists of two parts: (1) a primary, background wavefield that is unperturbed by the heterogeneity, and (2) a secondary, scattered wavefield that is generated at “sources” in the heterogeneities through scattering of the background wavefield.

Our discussion will closely follow section 13.2 of Aki and Richards (1980). We begin with the momentum equation for isotropic material (e.g., see equations 3.1 and 3.6 in the 227a notes).

$$\rho \ddot{u}_i = \partial_i(\lambda \partial_k u_k) + \partial_j[\mu(\partial_i u_j + \partial_j u_i)] \quad (8.1)$$

where \mathbf{u} is the displacement vector, ρ is density, and λ and μ are the Lamé parameters. At this point we are assuming a general inhomogeneous medium, so the partial derivatives on the r.h.s. will apply to λ and μ as well as to the displacement. Now assume that the inhomogeneous medium consists of the sum of two parts, an “unperturbed” homogeneous medium and the perturbations that make up the heterogeneity. Then the perturbed medium properties may be expressed as

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ \lambda &= \lambda_0 + \delta\lambda \\ \mu &= \mu_0 + \delta\mu \end{aligned} \quad (8.2)$$

where ρ_0 , λ_0 and μ_0 are for the unperturbed medium and are constant, and where $\delta\rho$, $\delta\lambda$ and $\delta\mu$ are the perturbations (functions of position but assumed to be much smaller than the unperturbed values). Substituting (8.2) into (8.1) we obtain

$$(\rho_0 + \delta\rho)\ddot{u}_i = \partial_i[(\lambda_0 + \delta\lambda)\partial_k u_k] + \partial_j[(\mu_0 + \delta\mu)(\partial_i u_j + \partial_j u_i)] \quad (8.3)$$

Now separate the homogeneous terms from the perturbed terms, remembering that the spatial derivatives of ρ_0 , λ_0 and μ_0 are zero.

$$\begin{aligned} \rho_0 \ddot{u}_i - \lambda_0 \partial_i \partial_k u_k - \mu_0 \partial_j (\partial_i u_j + \partial_j u_i) &= -\delta\rho \ddot{u}_i + \partial_i(\delta\lambda \partial_k u_k) + \partial_j[\delta\mu(\partial_i u_j + \partial_j u_i)] \\ \rho_0 \ddot{u}_i - (\lambda_0 + \mu_0) \partial_i \partial_k u_k - \mu_0 \partial_j \partial_j u_i &= -\delta\rho \ddot{u}_i + \delta\lambda \partial_i \partial_k u_k + (\partial_i \delta\lambda) \partial_k u_k + \delta\mu \partial_j \partial_j u_i \\ &\quad + \delta\mu \partial_j \partial_j u_i + (\partial_j \delta\mu)(\partial_i u_j + \partial_j u_i) \\ &= -\delta\rho \ddot{u}_i + (\delta\lambda + \delta\mu) \partial_i \partial_k u_k + \delta\mu \partial_j \partial_j u_i \\ &\quad + (\partial_i \delta\lambda) \partial_k u_k + (\partial_j \delta\mu)(\partial_i u_j + \partial_j u_i) \end{aligned} \quad (8.4)$$

We can also express this as

$$\begin{aligned} \rho_0 \ddot{u}_i - (\lambda_0 + \mu_0) \partial_i (\nabla \cdot \mathbf{u}) - \mu_0 \nabla^2 u_i &= -\delta \rho \ddot{u}_i + (\delta \lambda + \delta \mu) \partial_i (\nabla \cdot \mathbf{u}) + \delta \mu \nabla^2 u_i \\ &+ (\partial_i \delta \lambda) (\nabla \cdot \mathbf{u}) + (\partial_j \delta \mu) (\partial_i u_j + \partial_j u_i) \end{aligned} \quad (8.5)$$

where we have used $\partial_k u_k = \nabla \cdot \mathbf{u}$ and $\partial_j \partial_j = \nabla^2$. Now let us write the displacement \mathbf{u} as the sum of primary waves \mathbf{u}_0 and scattered waves \mathbf{u}_1

$$\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^1 \quad (8.6)$$

\mathbf{u}^0 is the solution for the unperturbed medium and so satisfies (8.5) with the r.h.s. set to zero

$$\rho_0 \ddot{u}_i^0 - (\lambda_0 + \mu_0) \partial_i (\nabla \cdot \mathbf{u}^0) - \mu_0 \nabla^2 u_i^0 = 0 \quad (8.7)$$

Now substitute (8.6) into (8.5) to obtain

$$\begin{aligned} \rho_0 (\ddot{u}_i^0 + \ddot{u}_i^1) - (\lambda_0 + \mu_0) \partial_i [\nabla \cdot (\mathbf{u}^0 + \mathbf{u}^1)] - \mu_0 \nabla^2 (u_i^0 + u_i^1) \\ = -\delta \rho (\ddot{u}_i^0 + \ddot{u}_i^1) + (\delta \lambda + \delta \mu) \partial_i [\nabla \cdot (\mathbf{u}^0 + \mathbf{u}^1)] + \delta \mu \nabla^2 (u_i^0 + u_i^1) \\ + (\partial_i \delta \lambda) [\nabla \cdot (\mathbf{u}^0 + \mathbf{u}^1)] + (\partial_j \delta \mu) [\partial_i (u_j^0 + u_j^1) + \partial_j (u_i^0 + u_i^1)] \end{aligned} \quad (8.8)$$

Notice that the \mathbf{u}_0 terms on the l.h.s. will sum to zero from (8.7). On the r.h.s. we drop the \mathbf{u}_1 terms since these are second order terms that involve products between the scattered waves (assumed small) and the medium perturbations (also assumed small). In other words, we consider only single scattering and neglect any higher order scattering. We then have

$$\begin{aligned} \rho_0 \ddot{u}_i^1 - (\lambda_0 + \mu_0) \partial_i (\nabla \cdot \mathbf{u}^1) - \mu_0 \nabla^2 u_i^1 &= -\delta \rho \ddot{u}_i^0 + (\delta \lambda + \delta \mu) \partial_i^0 (\nabla \cdot \mathbf{u}^0) + \delta \mu \nabla^2 u_i^0 \\ &+ (\partial_i \delta \lambda) (\nabla \cdot \mathbf{u}^0) + (\partial_j \delta \mu) (\partial_i u_j^0 + \partial_j u_i^0) \end{aligned} \quad (8.9)$$

Let us identify and define the r.h.s. as the local body force \mathbf{Q} so we have

$$\rho_0 \ddot{u}_i^1 - (\lambda_0 + \mu_0) \partial_i (\nabla \cdot \mathbf{u}^1) - \mu_0 \nabla^2 u_i^1 = Q_i \quad (8.10)$$

where

$$\begin{aligned} Q_i &= -\delta \rho \ddot{u}_i^0 + (\delta \lambda + \delta \mu) \partial_i^0 (\nabla \cdot \mathbf{u}^0) + \delta \mu \nabla^2 u_i^0 \\ &+ (\partial_i \delta \lambda) (\nabla \cdot \mathbf{u}^0) + (\partial_j \delta \mu) (\partial_i u_j^0 + \partial_j u_i^0) \end{aligned} \quad (8.11)$$

(8.10) is the equation of motion for the scattered wavefield \mathbf{u}^1 in a homogeneous isotropic medium with body force \mathbf{Q} that results from the local interaction of the heterogeneity with the primary wavefield \mathbf{u}^0 . Let us now see what form of \mathbf{Q} results when P or S plane waves are assumed as the primary wavefield.

Primary plane P-waves

Assume the waves are propagating in the x_1 direction. Then the u_2 and u_3 components of displacement are zero and we may write

$$u_i^0 = \delta_{1i} e^{-i\omega(t-x_1/\alpha_0)} \quad (8.12)$$

where $\alpha_0 = \sqrt{(\lambda_0 + 2\mu_0)/\rho_0}$ is the P velocity in the unperturbed medium. We may then express the temporal and spatial derivatives of \mathbf{u}^0 as

$$\begin{aligned} \ddot{u}_i^0 &= -\delta_{1i} \omega^2 u_1^0 \\ \nabla \cdot \mathbf{u}^0 &= \partial_1 u_1^0 \\ &= (i\omega/\alpha_0) u_1^0 \\ \partial_i (\nabla \cdot \mathbf{u}^0) &= -\delta_{1i} (\omega^2/\alpha_0^2) u_1^0 \\ \nabla^2 u_i^0 &= \delta_{1i} \partial_k \partial_k u_1^0 \\ &= -\delta_{1i} (\omega^2/\alpha_0^2) u_1^0 \\ \partial_i u_j^0 &= \delta_{1i} \delta_{1j} (i\omega/\alpha_0) u_1^0 \end{aligned} \quad (8.13)$$

Substituting into (8.11) we obtain the three components of \mathbf{Q}

$$\begin{aligned} Q_1 &= \left[\delta\rho\omega^2 - \frac{(\delta\lambda + 2\delta\mu)\omega^2}{\alpha_0^2} + i\frac{\omega}{\alpha_0} \partial_1(\delta\lambda) + 2i\frac{\omega}{\alpha_0} \partial_1(\delta\mu) \right] e^{-i\omega(t-x_1/\alpha_0)} \\ Q_2 &= i\frac{\omega}{\alpha_0} \partial_2(\delta\lambda) e^{-i\omega(t-x_1/\alpha_0)} \\ Q_3 &= i\frac{\omega}{\alpha_0} \partial_3(\delta\lambda) e^{-i\omega(t-x_1/\alpha_0)} \end{aligned} \quad (8.14)$$

Note that Q_2 and Q_3 are only excited by spatial gradients in λ . The first two terms in the expression for Q_1 may be related to the P velocity perturbation as follows

$$\begin{aligned} \delta\alpha &= \alpha - \alpha_0 \\ &= \sqrt{\frac{\lambda_0 + 2\mu_0 + \delta\lambda + 2\delta\mu}{\rho_0 + \delta\rho}} - \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \end{aligned} \quad (8.15)$$

For $x \gg dx$ and $y \gg dy$, we have the approximation

$$\frac{x + dx}{y + dy} = \frac{x}{y} \left(1 + \frac{dx}{x} - \frac{dy}{y} \right) \quad (8.16)$$

and thus we can express (8.15) as

$$\begin{aligned} \delta\alpha &= \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \left(1 + \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} - \frac{\delta\rho}{\rho_0} \right) - \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \\ &= \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \left[\sqrt{1 + \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} - \frac{\delta\rho}{\rho_0}} - 1 \right] \end{aligned} \quad (8.17)$$

Next, note that for $\epsilon \ll 1$, we have the approximation

$$\sqrt{1 + \epsilon} = 1 + \epsilon/2 \quad (8.18)$$

Thus, we can express (8.17) as

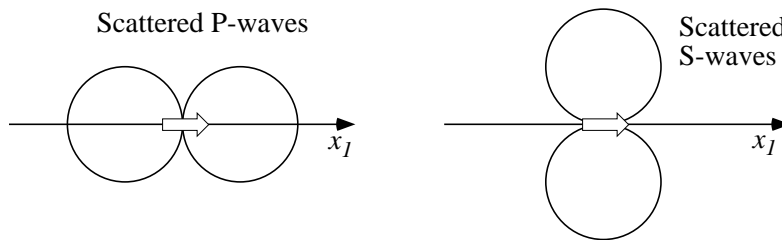
$$\begin{aligned} \delta\alpha &= \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \left[1 + \frac{1}{2} \frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} - \frac{1}{2} \frac{\delta\rho}{\rho_0} - 1 \right] \\ &= \frac{\alpha_0}{2} \left[\frac{\delta\lambda + 2\delta\mu}{\lambda_0 + 2\mu_0} - \frac{\delta\rho}{\rho_0} \right] \\ 2\frac{\delta\alpha}{\alpha_0} &= \frac{1}{\rho_0} \left[-\delta\rho + \frac{\rho_0(\delta\lambda + 2\delta\mu)}{\lambda_0 + 2\mu_0} \right] \\ -2\rho_0 \frac{\delta\alpha}{\alpha_0} &= \delta\rho - \frac{\delta\lambda + 2\delta\mu}{\alpha_0^2} \end{aligned} \quad (8.19)$$

Note that this is in a form that may be substituted for the first two terms of the expression for Q_1 in (8.14), e.g.

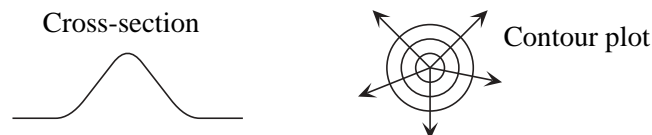
$$\delta\rho\omega^2 - \frac{(\delta\lambda + 2\delta\mu)\omega^2}{\alpha_0^2} = -2\omega^2\rho_0 \frac{\delta\alpha}{\alpha_0} \quad (8.20)$$

In this way, the dependence on $\delta\rho$, $\delta\lambda$ and $\delta\mu$ may be replaced with dependence on $\delta\alpha$, and we are left with only 3 independent parameters that determine the scattering. For the case of an incident P wave, the components of \mathbf{Q} are sensitive to perturbations in: (1) P velocity, (2) the gradient of $\delta\lambda$, and (3) the gradient of $\delta\mu$. Let us now explore what the far-field radiation of the scattered energy will look like in each case, assuming a localized perturbation small enough to be considered a point source.

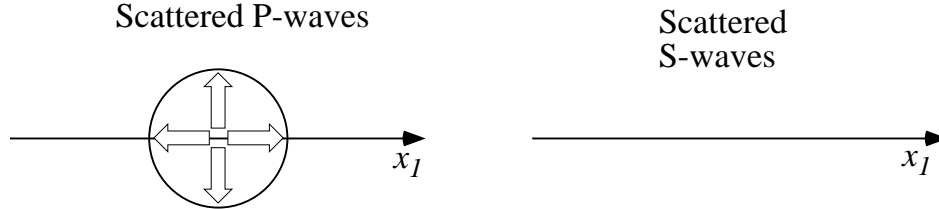
The P velocity perturbation term only enters into the x_1 component of \mathbf{Q} and acts as a single force in the x_1 direction. A small element of this source will generate scattered far-field P and S -waves with a radiation pattern that looks like:



Now consider the $\nabla(\delta\lambda)$ term. Let us imagine that we have a small blob of λ



The $\nabla(\delta\lambda)$ vectors point outward in all directions and the far-field radiation pattern will look like:



Note that this term acts like an explosive source, radiating P waves equally in all direction and generating no S waves. Recall the definition of the moment tensor in terms of body force equivalents (e.g., p. 55 of Aki and Richards)

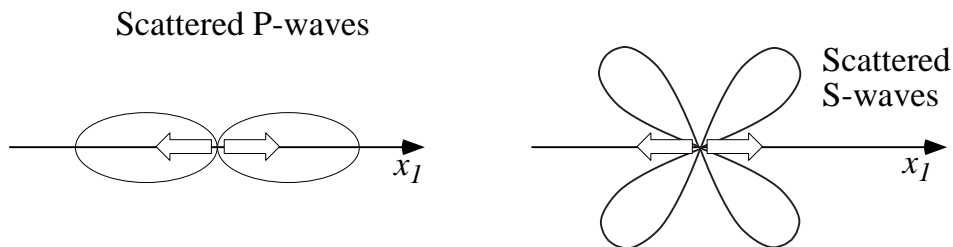
$$M_{pq} = \int_V f_p x_q dV(\mathbf{x}) \quad (8.21)$$

where \mathbf{f} is the body force vector and \mathbf{x} is the position within V . If $\delta\lambda$ is localized in a small region V , we then have

$$\int_V x_i \partial_k(\delta\lambda) dV = -\delta_{ik} \int_V \delta\lambda dV \quad (8.22)$$

and we see that the moment tensor is diagonal.

Finally, consider the $\partial_x(\delta\mu)$ term. This term only affects Q_1 and acts as a (1,1) dipole for a localized $\delta\mu$ anomaly, giving a far-field radiation pattern:



In this case the moment tensor only has a M_{11} element.

Note that in all three cases that we have considered, there is no scattered S energy in the exact direction of the plane P wave or back-scattered energy opposite to this direction. Note also that the scattering is frequency dependent with more scattering predicted at larger values of ω . Often we will replace the ω/α_0 factors in (8.14) with the wavenumber ($k = \omega/\alpha_0$); thus the $\delta\alpha$ scattering will scale as k^2 while the $\nabla(\delta\lambda)$ and $\nabla(\delta\mu)$ scattering scale with k .

Primary plane S-waves

Now let us consider the case of an incident S plane wave traveling in the x_1 direction with particle motion in the x_2 direction

$$u_i^0 = \delta_{2i} e^{-i\omega(t-x_1/\beta_0)} \quad (8.23)$$

where $\beta_0 = \sqrt{\mu_0/\rho_0}$ is the S velocity in the unperturbed medium. The temporal and spatial derivatives of \mathbf{u}^0 are

$$\begin{aligned} \ddot{u}_i^0 &= -\delta_{2i} \omega^2 u_2^0 \\ \nabla \cdot \mathbf{u}^0 &= \partial_1 u_2^0 \\ &= (i\omega/\beta_0) u_2^0 \\ \partial_i (\nabla \cdot \mathbf{u}^0) &= -\delta_{1i} (\omega^2/\beta_0^2) u_2^0 \\ \nabla^2 u_i^0 &= \delta_{2i} \partial_1 \partial_1 u_2^0 \\ &= -\delta_{2i} (\omega^2/\beta_0^2) u_2^0 \\ \partial_i u_j^0 &= \delta_{1i} \delta_{2j} (i\omega/\beta_0) u_2^0 \end{aligned} \quad (8.24)$$

Substituting into (8.10), we obtain

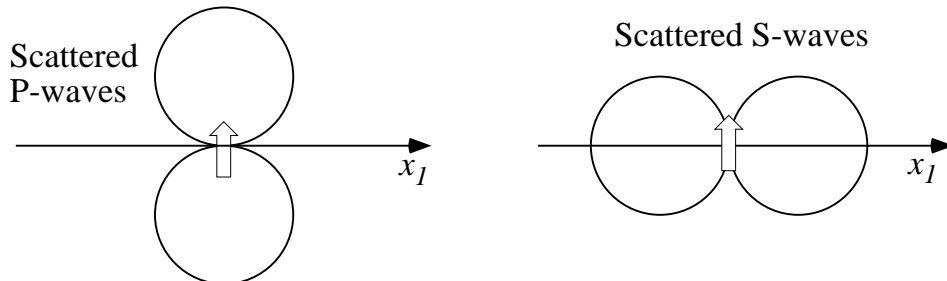
$$\begin{aligned} Q_1 &= i \frac{\omega}{\beta_0} \partial_2 (\delta\mu) e^{-i\omega(t-x_1/\beta_0)} \\ Q_2 &= \left[\delta\rho\omega^2 - \delta\mu \frac{\omega^2}{\beta_0^2} + i \frac{\omega}{\beta_0} \partial_1 (\delta\mu) \right] e^{-i\omega(t-x_1/\beta_0)} \\ Q_3 &= 0 \end{aligned} \quad (8.25)$$

As in the previous section, we can express the first two terms in the equation for Q_2 in terms of the velocity perturbation $\delta\beta$

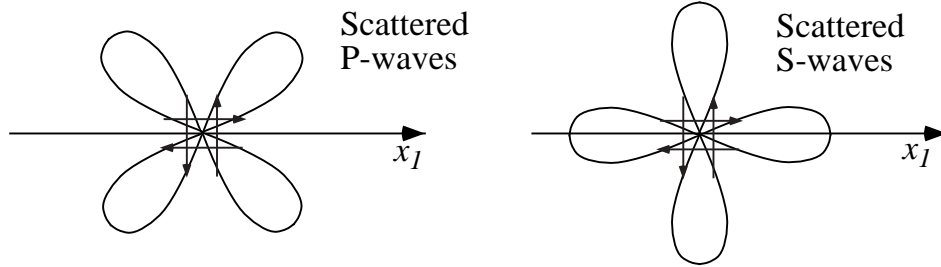
$$\delta\rho\omega^2 - \delta\mu \frac{\omega^2}{\beta^2} = -2\omega^2 \rho_0 \frac{\delta\beta}{\beta_0} \quad (8.26)$$

This may be derived by following (8.25–8.20), substituting the $\lambda + 2\mu$ terms with μ . Thus we see that the scattering from an incident S wave is sensitive to perturbations in β and in the spatial derivatives of μ . No scattering is caused by inhomogeneities in λ or its spatial derivatives.

Let us now consider the far-field radiation from small perturbations in β and μ . A localized anomaly in $\delta\beta$ will act as a single force in the x_2 direction and radiate both P and S energy:



The terms due to the spatial derivative of $\delta\mu$ correspond to a double couple when $\delta\mu$ is confined to a small region V . The moment tensor has nonvanishing elements $M_{12} = M_{21}$, proportional to $\int_V \delta\mu dV$



Note that, in this case, there is no scattered P energy in the direction of incident S propagation.

Wave equation solution for the scattered waves

In the previous section, we derived expressions for the body force \mathbf{Q} that is the effective source for the scattered waves. Now, let us write down solutions for the scattered wavefield, borrowing from results that we obtained earlier in the 227a class (i.e. solving the wave equation, seismic sources). Recall (8.10)

$$\rho_0 \ddot{u}_i^1 - (\lambda_0 + \mu_0) \partial_i (\nabla \cdot \mathbf{u}^1) - \mu_0 \nabla^2 u_i^1 = Q_i$$

As we showed in the wave equation derivation in 227a, this can be rewritten in the form

$$\rho_0 \ddot{\mathbf{u}}^1 - (\lambda_0 + 2\mu_0) \nabla (\nabla \cdot \mathbf{u}^1) + \mu_0 \nabla \times (\nabla \times \mathbf{u}^1) = \mathbf{Q} \quad (8.27)$$

By taking the divergence and curl of this equation, we can separate the P and S wave solutions and obtain

$$\nabla \cdot \ddot{\mathbf{u}}^1 - \alpha_0 \nabla^2 (\nabla \cdot \mathbf{u}^1) = \nabla \cdot \mathbf{Q} / \rho_0 \quad (8.28)$$

for the P waves and

$$\nabla \times \ddot{\mathbf{u}}^1 - \beta_0^2 \nabla^2 (\nabla \times \mathbf{u}^1) = \nabla \times \mathbf{Q} / \rho_0 \quad (8.29)$$

for the S waves. These have solutions

$$\nabla \cdot \mathbf{u}^1(\mathbf{x}, t) = \frac{1}{4\pi\alpha_0^2\rho_0} \int_V \frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} \nabla \cdot \mathbf{Q} \left(\boldsymbol{\xi}, t - \frac{|\mathbf{x} - \boldsymbol{\xi}|}{\alpha_0} \right) dV(\boldsymbol{\xi}) \quad (8.30)$$

and

$$\nabla \times \mathbf{u}^1(\mathbf{x}, t) = \frac{1}{4\pi\beta_0^2\rho_0} \int_V \frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} \nabla \times \mathbf{Q} \left(\boldsymbol{\xi}, t - \frac{|\mathbf{x} - \boldsymbol{\xi}|}{\beta_0} \right) dV(\boldsymbol{\xi}) \quad (8.31)$$

Notice that $1/|\mathbf{x} - \boldsymbol{\xi}|$ is our familiar $1/r$ geometrical spreading factor from the point of scattering to a receiver at \mathbf{x} and $|\mathbf{x} - \boldsymbol{\xi}|/c$ is simply the propagation time between the scattering point and receiver (where c is the P or S velocity).

Scattering due to velocity perturbation

Now consider the case where velocity varies in all directions with a finite scale length and we have a scalar P wave ($\Phi = \nabla \cdot \mathbf{u}$). Our simplest result will be obtained if we neglect the terms involving the spatial gradients in the medium properties; this approximation is valid if the inhomogeneities are smooth relative to the seismic wavelength. If we assume a plane wave propagating in the x_1 direction, then the primary wave has the form

$$\Phi^0 = A e^{-i\omega(t-x_1/c_0)} \quad (8.32)$$

where A is amplitude and c_0 is the unperturbed velocity. The solution for the scattered wavefield (8.30) requires $\nabla \cdot \mathbf{Q}$. We have from (8.14) and (8.20) that

$$Q_1 = -2A\omega^2 \rho_0 \frac{\delta c}{c_0} e^{-i\omega(t-x_1/c_0)}, \quad Q_2 = 0, \quad Q_3 = 0 \quad (8.33)$$

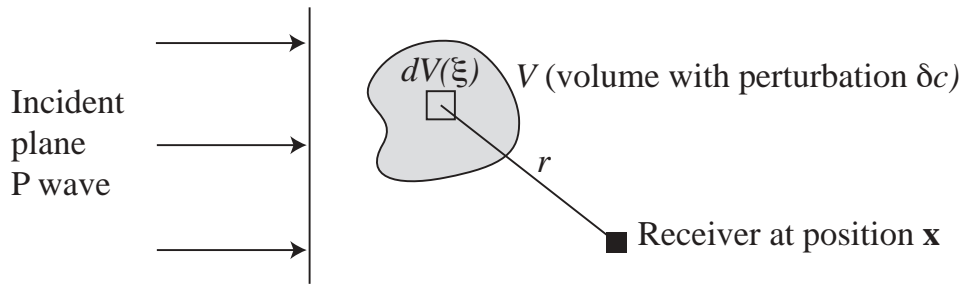
where we have dropped the $\partial(\delta\lambda)$ and $\partial(\delta\mu)$ terms. We thus have

$$\begin{aligned} \nabla \cdot \mathbf{Q} &= \frac{\partial}{\partial x_1} \left[-2A\omega^2 \rho_0 \frac{\delta c}{c_0} e^{-i\omega(t-x_1/c_0)} \right] \\ &\approx -2A\omega^2 \rho_0 \frac{\delta c}{c_0} i \frac{\omega}{c_0} e^{-i\omega(t-x_1/c_0)} \end{aligned} \quad (8.34)$$

where we again neglect the term involving the gradient of the velocity perturbation. Substituting into (8.30) we obtain

$$\begin{aligned} \Phi^1(\mathbf{x}, t) &= \nabla \cdot \mathbf{u}^1(\mathbf{x}, t) \\ &= \frac{1}{4\pi c_0^2 \rho_0} \int_V \frac{1}{|\mathbf{x} - \boldsymbol{\xi}|} \nabla \cdot \mathbf{Q} \left(\boldsymbol{\xi}, t - \frac{|\mathbf{x} - \boldsymbol{\xi}|}{c_0} \right) dV(\boldsymbol{\xi}) \\ &= \frac{A\omega^2}{2\pi c_0^2} \int_V -\frac{1}{r} \frac{\delta c}{c_0} e^{-i\omega(t-r/c_0-\xi_1/c_0)} dV(\boldsymbol{\xi}) \end{aligned} \quad (8.35)$$

where $r = |\mathbf{x} - \boldsymbol{\xi}|$ and V is the region where $\delta c \neq 0$.



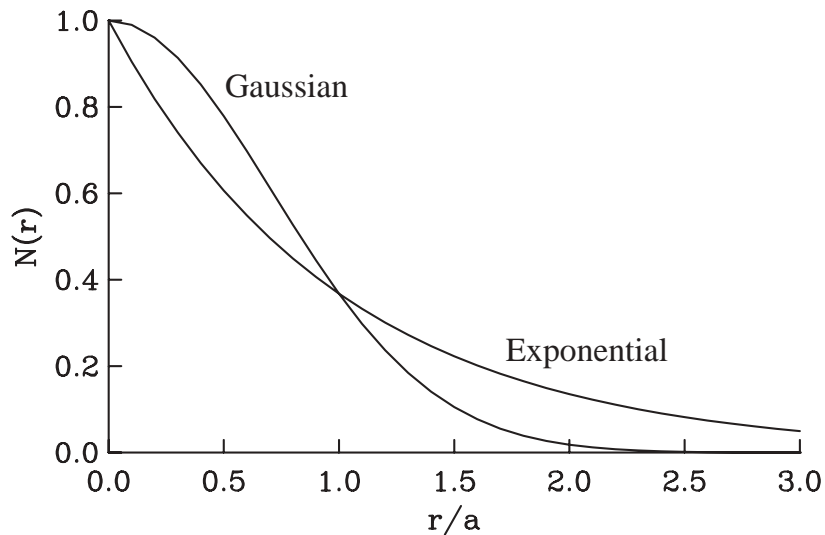
This equation could be used in a computer program if one actually knew δc everywhere in the scattering volume of interest. However, normally one has no hope of actually resolving all of the individual scatterers but only some statistical measure of their scale length and strength. A standard way to describe the spatial fluctuation of a random field is with the autocorrelation function. Let us define the fractional velocity perturbation as:

$$\mu = -\delta c/c_0 \quad (8.36)$$

(do not confuse this parameter with the shear modulus!) where we assume the fluctuation of μ is isotropic and stationary in space. The normalized autocorrelation function is

$$N(\mathbf{r}) = \frac{\langle \mu(\mathbf{r}')\mu(\mathbf{r}' + \mathbf{r}) \rangle}{\langle \mu^2 \rangle} \quad (8.37)$$

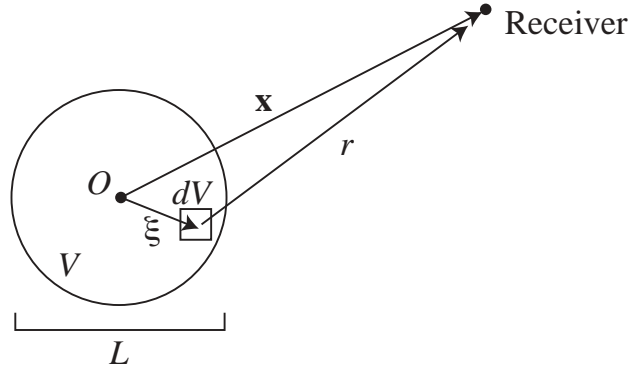
where $\langle \rangle$ is a spatial average over many statistically independent samples. Two specific forms for $N(\mathbf{r})$ are often modeled:



$$\begin{aligned} N(\mathbf{r}) &= e^{-|\mathbf{r}|/a} && \text{(exponential model)} \\ &= e^{-|\mathbf{r}|^2/a^2} && \text{(Gaussian model)} \end{aligned} \quad (8.38)$$

where a is called the *correlation distance*. Note that the Gaussian model will have “blobs” of relatively uniform size, whereas the exponential model will have greater heterogeneity structure at both smaller and larger wavelengths.

Now let us consider the scattered waves at a distance far away from an inhomogeneous region confined in a small volume V with linear dimension L .



In order to evaluate the integral in (8.35), we approximate the scatter-to-receiver distance as:

$$\begin{aligned} r &= (|\mathbf{x}|^2 + |\boldsymbol{\xi}|^2 - 2\mathbf{x} \cdot \boldsymbol{\xi})^{1/2} \\ &\approx |\mathbf{x}| - \hat{\mathbf{n}} \cdot \boldsymbol{\xi} \end{aligned} \quad (8.39)$$

where $\hat{\mathbf{n}}$ is the unit vector in the direction of \mathbf{x} . Note that the first (exact) expression follows from the law of cosines and the dot product definition. This approximation is valid provided:

$$\frac{kL^2}{2|\mathbf{x}|} \ll \frac{\pi}{2} \quad (8.40)$$

where k is the wavenumber. Recalling that $k = \omega/c = 2\pi/\Lambda$ where Λ is the wavelength, this condition is equivalent to

$$\frac{L^2}{\Lambda|\mathbf{x}|} \ll \frac{1}{2} \quad (8.41)$$

and we see this is a far-field approximation that is valid provided the wavelength and distance are large enough compared to the size of the volume heterogeneity.

Putting (8.39) into (8.35), replacing $1/r$ with $1/|\mathbf{x}|$, using $\mu = \delta c/c_0$, and setting $k = \omega/c_0$, we have

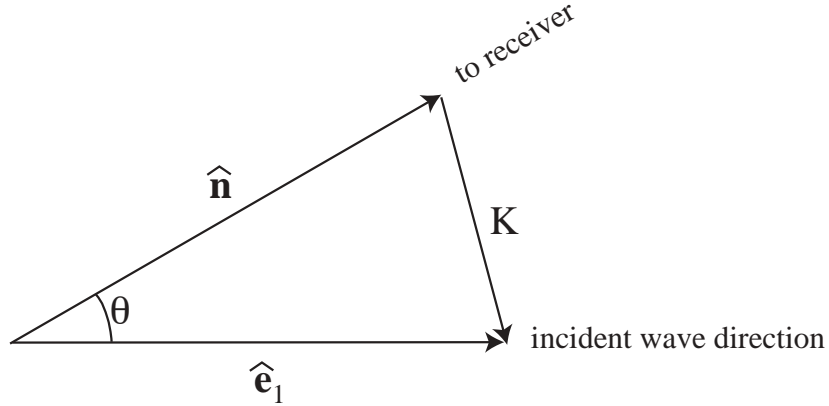
$$\begin{aligned} \Phi^1(\mathbf{x}, t) &= \frac{A\omega^2}{2\pi c_0^2} \int_V -\frac{1}{r} \frac{\delta c}{c_0} e^{-i\omega(t-r/c_0-\xi_1/c_0)} dV(\boldsymbol{\xi}) \\ &= \frac{Ak^2}{2\pi|\mathbf{x}|} e^{-i(\omega t - k|\mathbf{x}|)} \int_V \mu(\boldsymbol{\xi}) e^{ik(\xi_1 - \hat{\mathbf{n}} \cdot \boldsymbol{\xi})} dV(\boldsymbol{\xi}) \end{aligned} \quad (8.42)$$

If we know only the statistical properties of $\mu(\mathbf{x})$ rather than its exact form, we cannot expect to evaluate this expression and obtain individual wiggles on a synthetic seismogram. Fortunately, however, a solution is possible if we consider only the power carried by the

scattered waves. The power is proportional to $|\Phi^1|^2$. Since $|\Phi^1|^2$ is equal to the product of Φ^1 and its complex conjugate, we have

$$|\Phi^1|^2 = \frac{A^2 k^4}{4\pi^2 |\mathbf{x}|^2} \int_V \int_V \mu(\xi') \mu(\xi) e^{ik[\xi_1 - \xi'_1 - \hat{\mathbf{n}} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}')] } dV(\boldsymbol{\xi}) dV(\boldsymbol{\xi}') \quad (8.43)$$

Now define $\hat{\mathbf{e}}_1$ as the unit vector in the ξ_1 direction (the direction of the incident wave) and θ as the scattering angle (the angle between the incident wave direction, $\hat{\mathbf{e}}_1$, and the scattered wave direction, $\hat{\mathbf{n}}$).



We define $\mathbf{K} = \hat{\mathbf{e}}_1 - \hat{\mathbf{n}}$. From the isosceles triangle in this figure, it is easily seen that $\sin(\theta/2) = |\mathbf{K}|/2$, since $|\hat{\mathbf{n}}| = |\hat{\mathbf{e}}_1| = 1$, and hence $|\mathbf{K}| = 2 \sin(\theta/2)$. Note that this definition of \mathbf{K} can be used to simplify part of (8.43):

$$\begin{aligned} \xi_1 - \xi'_1 - \hat{\mathbf{n}} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}') &= \hat{\mathbf{e}}_1 \cdot \boldsymbol{\xi} - \hat{\mathbf{n}} \cdot \boldsymbol{\xi} + \hat{\mathbf{e}}_1 \cdot \boldsymbol{\xi}' - \hat{\mathbf{n}} \cdot \boldsymbol{\xi}' \\ &= (\hat{\mathbf{e}}_1 - \hat{\mathbf{n}}) \cdot \boldsymbol{\xi} - (\hat{\mathbf{e}}_1 - \hat{\mathbf{n}}) \cdot \boldsymbol{\xi}' \\ &= \mathbf{K} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}') \end{aligned} \quad (8.44)$$

Provided our integration volume is large enough to fully sample the heterogeneity, taking the statistical average of (8.43) using (8.37) we have

$$\langle |\Phi^1|^2 \rangle = \frac{A^2 k^4 \langle \mu^2 \rangle}{4\pi^2 |\mathbf{x}|^2} \int_V \int_V N(\boldsymbol{\xi} - \boldsymbol{\xi}') e^{ik\mathbf{K} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}')} dV(\boldsymbol{\xi}) dV(\boldsymbol{\xi}') \quad (8.45)$$

To evaluate this integral we change the variables $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ to the relative coordinate $\hat{\boldsymbol{\xi}} = \boldsymbol{\xi} - \boldsymbol{\xi}'$ and the center-of-mass coordinate $\bar{\boldsymbol{\xi}} = (\boldsymbol{\xi} + \boldsymbol{\xi}')/2$ and obtain

$$\begin{aligned} \langle |\Phi^1|^2 \rangle &= \frac{A^2 k^4 \langle \mu^2 \rangle}{4\pi^2 |\mathbf{x}|^2} \int_V \int_V N(\hat{\boldsymbol{\xi}}) e^{ik\mathbf{K} \cdot \hat{\boldsymbol{\xi}}} dV(\hat{\boldsymbol{\xi}}) dV(\bar{\boldsymbol{\xi}}) \\ &= \frac{A^2 k^4 \langle \mu^2 \rangle V}{4\pi^2 |\mathbf{x}|^2} \int_V N(\hat{\boldsymbol{\xi}}) e^{ik\mathbf{K} \cdot \hat{\boldsymbol{\xi}}} dV(\hat{\boldsymbol{\xi}}) \\ &= \frac{A^2 k^4 \langle \mu^2 \rangle V}{4\pi^2 |\mathbf{x}|^2} \int_V N(\hat{\boldsymbol{\xi}}) e^{ik\mathbf{K} \cdot \hat{\boldsymbol{\xi}}} d\hat{\xi}_1 d\hat{\xi}_2 d\hat{\xi}_3 \end{aligned} \quad (8.45)$$

where $V = \int_V dV(\bar{\xi})$.

Next, we change from $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3)$ to the spherical coordinates (r', θ', ϕ') , with \mathbf{K} as the polar axis, obtaining:

$$\begin{aligned} r' &= |\hat{\xi}| \\ \mathbf{K} \cdot \hat{\xi} &= |\mathbf{K}| r' \cos \theta' \\ d\hat{\xi}_1 d\hat{\xi}_2 d\hat{\xi}_3 &= r'^2 dr' \sin \theta' d\theta' d\phi' \end{aligned} \quad (8.46)$$

We then obtain

$$\begin{aligned} \int_V N(\hat{\xi}) e^{ik\mathbf{K} \cdot \hat{\xi}} d\hat{\xi}_1 d\hat{\xi}_2 d\hat{\xi}_3 &= \int_V N(r') e^{ik|\mathbf{K}| r' \cos \theta'} r'^2 dr' \sin \theta' d\theta' d\phi' \\ &= 4\pi \int_0^\infty N(r') \frac{\sin(k|\mathbf{K}|r')}{k|\mathbf{K}|} r' dr' \end{aligned} \quad (8.47)$$

where the integration limit for r' is extended to infinity, assuming that the correlation distance a is much smaller than the linear dimension of V .

This integral can be evaluated for the cases $N(r) = e^{-r/a}$ (exponential) and $N(r) = e^{-r^2/a^2}$ (Gaussian), and the result put into (8.45):

$$\langle |\Phi^1|^2 \rangle = \frac{2A^2 k^4 \langle \mu^2 \rangle a^3 V}{\pi |\mathbf{x}|^2} \frac{1}{(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2})^2} \quad \text{for } N(r) = e^{-r/a} \quad (8.48)$$

and

$$\langle |\Phi^1|^2 \rangle = \frac{A^2 k^4 \langle \mu^2 \rangle a^3 V}{4\sqrt{\pi} |\mathbf{x}|^2} e^{-k^2 a^2 \sin^2 \frac{\theta}{2}} \quad \text{for } N(r) = e^{-r^2/a^2} \quad (8.49)$$

In both cases the power of the scattered waves is proportional to k^4 when $ka \ll 1$. This is termed *Rayleigh scattering*. If ka is small, the scattered power does not depend upon the scattering angle θ . Thus velocity perturbations with scale length much smaller than a wavelength produce isotropic scattering. However, when ka is small, the gradients of velocity and elasticity perturbation (neglected so far in our analysis) become important and their effects are directional.

When ka is large the scattering due to velocity perturbation is mostly directed forward and the scattered power is concentrated within an angle $(ka)^{-1}$ around the direction of primary wave propagation ($\theta = 0$). Back-scattered power ($\theta = \pi$) becomes very small, particularly for the Gaussian model.

A more complete scattering model can be derived by taking into account the gradients in elastic properties. If we assume that the medium behaves like an Poisson solid (this provides the scaling between the P and S-wave velocity perturbations), then for the exponential autocorrelation model, one can show that the average scattered power is given by:

$$\langle |\Phi^1|^2 \rangle = \frac{2A^2 k^4 a^3 \langle \mu^2 \rangle V}{\pi r^2} \frac{\frac{1}{4} (\cos \theta + \frac{1}{3} + \frac{2}{3} \cos^2 \theta)^2}{(1 + 4k^2 a^2 \sin^2 \frac{\theta}{2})^2} \quad (8.50)$$

where r is the scattering receiver distance. Note that $\langle \mu^2 \rangle$ is simply the square of the RMS velocity perturbation ($\mu = \delta c/c_0$) of the random medium. This equation does, however, neglect the effect of density perturbations. Density perturbations tend to increase the amount of backscattered energy. If these are important, then still more complete equations need to be used. In addition, in some cases P-to-S and S-to-P scattering should also be included. A good general reference for scattering theories that includes all of these complications is the textbook by Sato and Fehler (1998).

How To Write a Born Scattering Program

Most scattering programs are based on ray theory so you will need to be able to trace rays through your model and to compute travel time and geometrical spreading factors.

- (1) Define the background velocity vs. depth model, the source and receiver locations, and the ray paths to be modeled.
- (2) Decide on what type of random media (e.g., exponential, Gaussian, etc.) and what scattering equation you will use (e.g., 8.48, 8.50, etc.). This will determine what parameters you will need to specify the scattering part of the model. Determine the frequency (ω) at which you will model the scattering.
- (3) Determine where the scattering volume is in the Earth that you will use to model your observations. Specify the heterogeneity parameters that you will need, such as the scale length, the RMS velocity heterogeneity, the P-to-S scaling, etc.
- (4) Divide the scattering volume into cells that you will use to numerically integrate the scattered power.
- (5) For each source-receiver pair, initialize a time series to zero values.
- (6) For each cell in your scattering volume, compute the source-to-cell travel time and amplitude, A , of the incident wave. Compute the scattering angle, θ , the difference between the incident ray direction and the takeoff direction of the scattered ray that will land at the receiver (this is one of the trickier parts so be sure to thoroughly test this part of the code!). Compute the geometrical spreading factor for the scattered ray (this will replace the $1/r^2$ factor (8.49), etc.). Compute the local wavenumber k from ω and the average velocity in the cell.
- (7) Use your preferred scattering equation to compute the amount of scattered power that will arrive at the receiver. Using the total source-to-scatterer-to-receiver travel time, add this contribution to your time series.
- (8) Repeat (6) and (7) for all the cells in your scattering volume.
- (9) Repeat (5)-(8) for all of your source-receiver pairs.
- (10) Your synthetics will give power as a function of time. If they are noisy looking, try using a longer sample interval dt for your time series or convolve the result with a realistic source-time function (in energy, not amplitude!).
- (11) Take the square root if you want the amplitude envelopes.

- (12) Often you will want to compare the scattered power to that in the direct arrival. To do so, simply compute the ray theoretical amplitude for each source-to-receiver ray path.
- (13) You can add in the effect of Q along the ray paths and reflection and transmission coefficients where the rays cross boundaries if you want to include these effects.

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