Secondary sources (Huygens and Kirchhoff)

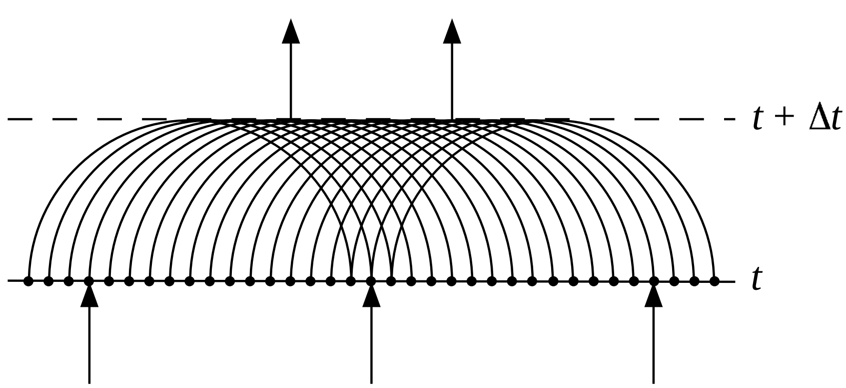
So far we have concentrated on synthetic seismogram calculations for the case of one-dimensional Earth models in which the velocity varies only as a function of depth. Under this assumption, we have shown how ray theoretical methods such as WKBJ and matrix formulations such as reflectivity can be used to solve the wave equation. Our treatment was based on a flat earth, but can also be used for a radially symmetric earth by applying the earth flattening transformation.

To a good first order approximation, the deep earth is close to spherically symmetric so these methods often are adequate for modeling observed seismograms. However, lateral heterogeneity is always present to some degree, particularly in the crust, and is often the target of greatest interest in seismic studies now that the average radial velocity structure has been determined. Computing synthetic seismograms in 3D velocity structures is much more complicated than the 1D calculation. Ray theoretical methods can be generalized to 3D (e.g. Maslov or Gaussian beam methods), but the ray tracing can be tricky and the results still suffer from the limitations of ray theory. Reflectivity methods cannot be generalized to 3D. Finite differences provide exact solutions in 3D, but at great computational cost.

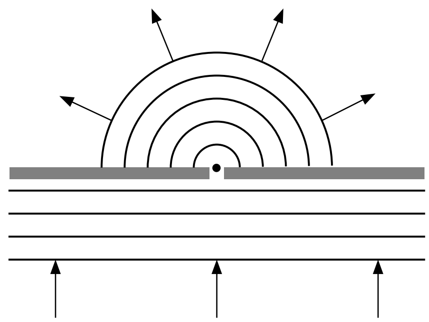
In this section, we present some methods for computing synthetic seismograms that are very useful for certain types of laterally heterogeneous models. They cannot be used in every case, but when applicable, they often can produce accurate synthetics with good computational efficiency. These techniques all involve the concept that each point on the wavefront can be considered to generate a secondary source, and that the response at a receiver can be computing by summing the contributions from the secondary sources.

Huygens' principle

This idea was first described by Huygens (c. 1678) and is often called Huygens' principle. It is most commonly mentioned in the context of light waves and optical ray theory, but is applicable to any wave propagation problem. If we consider a plane wavefront traveling in a homogeneous medium, we can see how the wavefront can be thought to propagate through the constructive interference of secondary wavelets:



This simple idea provides, at least in a qualitative sense, an explanation for the behavior of waves when they pass through a narrow aperture:

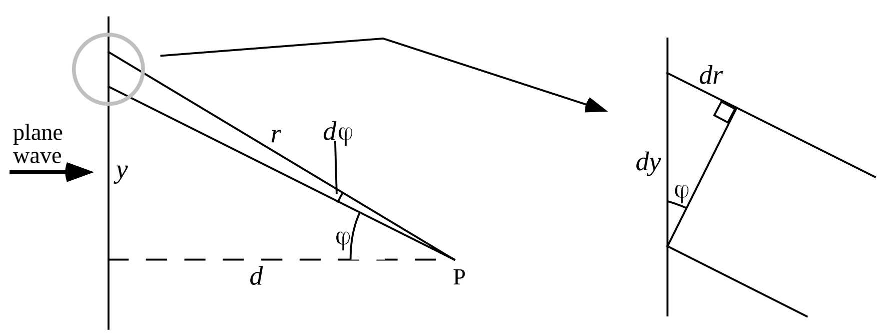


The bending of the ray paths at the edges of the gap is termed *diffraction.* The degree to which the waves diffract into the "shadow" of the obstacle depends upon the wavelength of the waves in relation to the size of the opening. At relatively long wavelengths (e.g. ocean waves striking a hole in a jetty), the transmitted waves will spread out almost uniformly over 180°. However, at short wavelengths the diffraction from the edges of the slot will produce a much smaller spreading in the wavefield. For light waves, very narrow slits are required to produce noticeable diffraction. These properties can be modeled using Huygens' principle by computing the effects of constructive and destructive intererence at different wavelengths.

Huygens' principle is a useful concept since it provides a simple way to gain an intuitive understanding of many aspects of wave behavior. However, it fails as a quantitative theory in several respects: (1) it says nothing about what amplitude the secondary waves should have, or how their "radiation pattern" might vary as a function of ray angle, (2) it predicts the wrong phase for the secondary arrivals, (3) it does not explain why the waves should not radiate backwards.

A simple plane wave example

To understand this better, let's attempt to use Huygens' principle to model plane wave propagation. Consider a receiver P located a distance d in front of a plane wave of amplitude A traveling at velocity c in a homogeneous whole space:



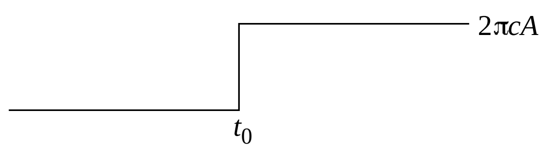
The current position of the wavefront is specified as $t = 0$. Define $t_0 = d/c$ as the time that the wavefront will pass by P. Now, let us sum the contributions from the spherical wavefronts generated by each point on the wavefront at time t ($t > t_0$). Waves that arrive within a time interval dt are from a ring on the surface at distance r from P. This ring has a radius y and width dy. The surface area of the ring, dS, may be expressed as

$$\begin{aligned} dS &= 2\pi y\, dy \\ &= 2\pi(r \sin\theta)(dr/\sin\theta) \\ &= 2\pi r\, dr \\ &= 2\pi rc\, dt \end{aligned} \qquad (hk.1)$$

The expected amplitude at P is then given by multiplying the area dS by the amplitude of the incident plane wave A and the geometrical spreading factor for spherical waves, $1/r$, and dividing by the time interval dt.

$$\begin{aligned} A_P(t > t_0) &= A\, dS(1/r)(1/dt) \\ &= A2\pi rc\, dt(1/r)(1/dt) \\ &= 2\pi cA \end{aligned} \qquad (hk.2)$$

Notice that A_P has the form of a step function with height $2\pi cA$.



Now imagine that the plane wave has a source-time function $A = f(t)$. We would like to form the response at P as a convolution between $f(t)$ and the result of our Huygens calculation. For a plane wave in a homogeneous whole space we already know the answer that we should get—a delta function at t_0.

$$A(t_0) = f(t) * \delta(t_0) \quad \text{desired result}$$

The pulse should travel to P unchanged in both amplitude and shape. Yet our calulation predicts that the convolutional operator has the form of a step function with height $2\pi c$

$$A(t_0) = f(t) * 2\pi cH(t_0) = f(t) * G(t) \quad \text{Huygens result}$$

where H is the Heaviside unit step function, and $G(t) \equiv 2\pi cH(t_0)$ is the result of our Huygens' calculation. Since the derivative of $H(t)$ is $\delta(t)$ we can fudge our solution into the correct form by taking the derivative and dividing by $2\pi c$

$$\begin{aligned} A(t_0) &= f(t) * (\frac{1}{2\pi c})\frac{\partial}{\partial t}G(t) \\ &= \frac{1}{2\pi c} f'(t) * G(t) \end{aligned} \qquad (hk.3)$$

where $f' = \frac{\partial f}{\partial t}$ and we have used $\frac{\partial}{\partial t}[f(t) * g(t)] = f'(t) * g(t) = f(t) * g'(t)$.

Is there a simple explanation for where these additional terms might come from? Some insight may be gained from the expression for the far-field radiation from a point source (eqn. 7.12 from the Introduction to Seismology text)

$$u(r,t) = \left(\frac{1}{rc}\right) \frac{\partial f(t - r/c)}{\partial t} \qquad (hk.4)$$

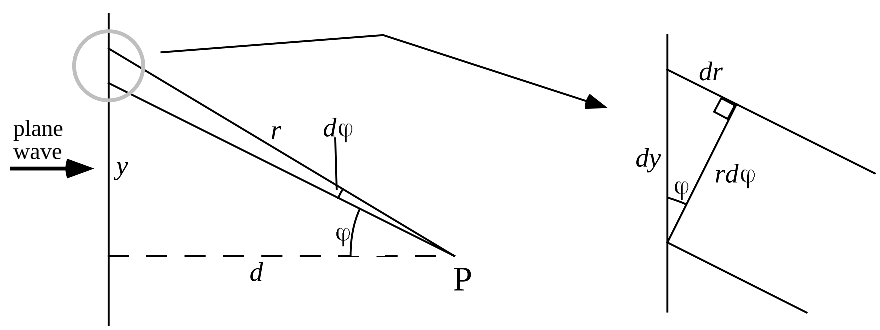
This provides some rationale for the $1/c$ factor in (hk.3) and for using the time derivative of $f(t)$ as our effective source-time function for the Huygens wavelets if we assume that we are in the far-field. However, we are still left with a factor of $1/2\pi$ that is unexplained. Although we can scale the Huygens' result to produce the correct answer in this particular case, we have no guarantee that this will work for other situations, nor do we understand where these correction factors come from.

Kirchhoff theory

(Introduction to Seismology (ITS) textbook, p. 133–138, eqn. 7.34–7.66)

The plane wave example revisited

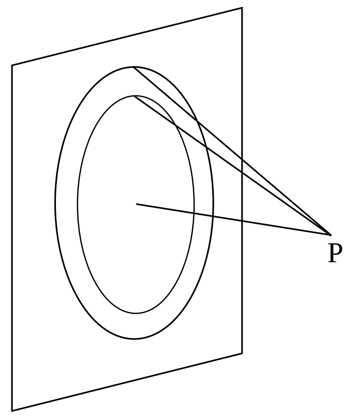
Before continuing, let us now try out our new formalism on the simple plane wave example that we earlier attempted to solve using Huygens' principle. To model a plane wave, we let $r_0 \to \infty$ and assume that the wave has unit amplitude on S at time $t = 0$



To obtain an exact solution, let us use ITS 7.65, containing both the near-field and far-field terms. The $1/r_0$ geometrical spreading term is not needed in the case of a plane wave, but the $1/r_0^2$ term goes to zero and $\cos\theta_0 = -1$ since the rays are perpendicular to dS. Thus we have

$$\begin{aligned}\phi_P(t) = & \frac{1}{4\pi}\int_S \delta(t - r/c)\frac{\cos\theta}{r^2} dS * f(t) \\ & + \frac{1}{4\pi}\int_S \delta(t - r/c)\frac{1+\cos\theta}{cr} dS * f'(t)\end{aligned} \qquad (hk.5)$$

Although (hk.5) was derived assuming a closed surface around P, it will be sufficient to evaluate the integral only over the plane, since we can imagine the curve being closed far enough away from P that its contributions would arrive later than any time of interest. As before, to evaluate the integral we define a ring with surface area dS



Recall (hk.1) for the area of the ring, $dS = 2\pi rc\,dt$, and note that $\cos\theta = d/r$. We thus have for the first term in (hk.5)

$$\begin{aligned}\frac{1}{4\pi}\int_S \delta(t-r/c)\frac{\cos\theta}{r^2}dS &= \frac{1}{4\pi}\frac{d/r}{r^2}(2\pi rc)H(t-d/c)\\ &= \frac{dc}{2r^2}H(t-d/c)\\ &= \frac{dc}{2c^2t^2}H(t-d/c)\\ &= \frac{d}{2ct^2}H(t-d/c)\end{aligned} \qquad (hk.6)$$

where $H(t-d/c)$ is the Heaviside step function and we have used $r = ct$. Similarly, the second term in (hk.5) becomes

$$\begin{aligned}\frac{1}{4\pi}\int_S \delta(t-r/c)\frac{1+\cos\theta}{cr}dS &= \frac{1}{4\pi}\frac{1+d/r}{cr}(2\pi rc)H(t-d/c)\\ &= \frac{1}{2}(1+d/r)H(t-d/c)\\ &= \frac{1}{2}(1+d/ct)H(t-d/c)\end{aligned} \qquad (hk.7)$$

Substituting (hk.6) and (hk.7) into (hk.5), we have

$$\phi_P(t) = \frac{d}{2ct^2}H(t-d/c)*f(t) + \frac{1}{2}\left(1+\frac{d}{ct}\right)H(t-d/c)*f'(t) \qquad (hk.8)$$

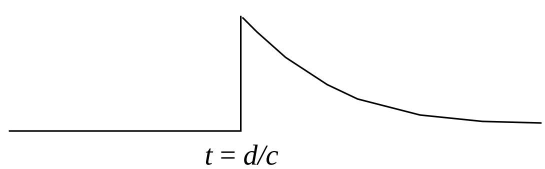
Moving the time derivative to the other side of the convolution, we obtain

$$\begin{aligned}
\phi_P(t) &= \frac{d}{2ct^2} H(t-d/c) * f(t) + \frac{\partial}{\partial t}\left[\left(\frac{1}{2} + \frac{d}{2ct}\right) H(t-d/c)\right] * f(t) \\
&= \frac{d}{2ct^2} H(t-d/c) * f(t) + \left[-\frac{d}{2ct^2} H(t-d/c) + \left(\frac{1}{2} + \frac{d}{2ct}\right)\frac{\partial}{\partial t} H(t-d/c)\right] * f(t) \\
&= \left(\frac{1}{2} + \frac{d}{2ct}\right)\delta(t-d/c) * f(t) \\
&= \left(\frac{1}{2} + \frac{d}{2cd/c}\right)\delta(t-d/c) * f(t) \\
&= \delta(t-d/c) * f(t) \\
&= f(t-d/c)
\end{aligned} \qquad (hk.9)$$

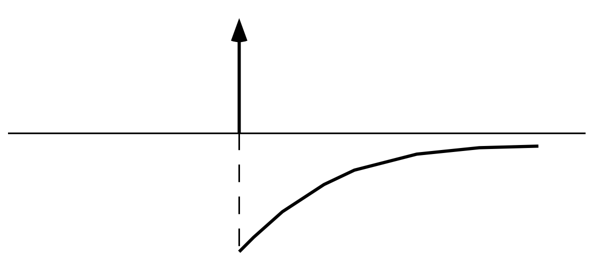
This is what we expected to obtain for the plane wave. The source time function is delayed by the travel time to the point P but the amplitude and wave shape are unchanged. Unlike the simple calculation based on Huygens' principle that we showed earlier, the Kirchhoff formula provides an exact solution without requiring any fudge factors. Note that an approximate solution may also be obtained by considering only the far-field term from (hk.5)

$$\phi_P(t) = \frac{1}{2}\left(1 + \frac{d}{ct}\right) H(t-d/c) * f'(t) \qquad (hk.10)$$

At $t = d/c$ this function steps from zero to one and then slowly decays as t increases:



The derivative of this function is $\delta(t-d/c)$ with a growing negative amplitude tail.



The delta function will dominate the response except at relatively long periods, where it becomes necessary to include the near-field term to cancel the effect of this tail.

Kirchhoff methods would not be very useful if they were only used to compute simple examples like this where we already know the answer. Their advantages come from the fact that they remain valid when the integration surface dS or the incident wavefield becomes more complicated. In these cases analytical solutions are generally impossible and the integral must be evaluated numerically.

Kirchhoff applications

(Introduction to Seismology, p. 138–141)

References

Kampmann, W. and G. Müller, PcP amplitude calculations for a core-mantle boundary with topography, *Geophys. Res. Lett.*, **16**, 653–656, 1989.

Longhust, R.S., *Geometrical and Physical Optics*, John Wiley and Sons, New York, 1967.

Scott, P. and D. Helmberger, Applications of the Kirchhoff-Helmholtz integral to problems in seismology, *Geophys. J. Roy. Astron. Soc.*, **72**, 237–254, 1985.