Reynold's transport theorem

Start with the most general theorem, which is Reynold's transport theorem for a fixed control volume.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\phi d\Omega = \frac{\partial}{\partial t}\int_\Omega \rho\phi d\Omega + \int_S \rho\phi\underline{\boldsymbol{u}}\cdot\hat{n}dS \tag{1}$$

the LHS is the total change of ϕ for a control volume which equals *the time rate of change of ϕ inside the control volume* plus *the net flux of ϕ through the control volume.*

Conservation of Mass

In order to consider the Conservation of mass, set $\phi = 1$, which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho d\Omega = \frac{\partial}{\partial t}\int_\Omega \rho d\Omega + \int_S \rho\underline{\boldsymbol{u}}\cdot\hat{n}dS \tag{2}$$

the LHS must equal zero due to conservation of mass. The $\frac{\partial}{\partial t}$ can go inside the integral because Ω doesn't depend on t and we can use Green's theorem (Divergence theorem) to convert the surface integral into a volume integral. Now the equation becomes

$$0 = \int_\Omega \frac{\partial}{\partial t}\rho d\Omega + \int_\Omega \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}})d\Omega = \int_\Omega \left(\frac{\partial\rho}{\partial t} + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}})\right)d\Omega \tag{3}$$

Now because Ω can be any arbitrary control volume, the expression inside the parentheses must be always true so we can drop the integral. We now have an equation for mass in a compressible fluid where ρ is not assumed to be uniform

$$0 = \frac{\partial\rho}{\partial t} + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}) \tag{4}$$

We can use the product rule to expand the $\left(\frac{\partial\rho}{\partial t} + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}})\right)$ term into two terms, $\underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{\nabla}}\rho + \rho\underline{\boldsymbol{\nabla}}\cdot\underline{\boldsymbol{u}}$. Rewriting we get

$$0 = \frac{\partial\rho}{\partial t} + \underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{\nabla}}\rho + \rho\underline{\boldsymbol{\nabla}}\cdot\underline{\boldsymbol{u}} \tag{5}$$

And we notice the material derivative $\frac{D}{Dt} = \frac{\partial\rho}{\partial t} + \underline{\boldsymbol{u}}\cdot\underline{\boldsymbol{\nabla}}$ will give us

$$0 = \frac{D\rho}{Dt} + \rho\underline{\boldsymbol{\nabla}}\cdot\underline{\boldsymbol{u}} \quad\text{or more simply}\quad 0 = \frac{D\rho}{Dt} + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}) \tag{6}$$

This is still the general case for a fluid that can have a spatially and time varying density. Now we assume the special case of incompressible flow in which ρ is uniform. The material derivative term vanishes and the expression reduces to

$$\underline{\boldsymbol{\nabla}}\cdot\underline{\boldsymbol{u}} = 0 \tag{7}$$

This demonstrates that in an incompressible flow, the divergence of velocity must be zero. These two notions are used interchangeably and referred to as the incompressibility constraint.

Conservation of Momentum

In order to consider the Conservation of momentum, set $\phi = \underline{\boldsymbol{u}}$, which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\underline{\boldsymbol{u}}d\Omega = \frac{\partial}{\partial t}\int_\Omega \rho\underline{\boldsymbol{u}}d\Omega + \int_S \rho\underline{\boldsymbol{u}}\left(\underline{\boldsymbol{u}}\cdot\hat{n}\right)dS \tag{8}$$

This time the LHS are the total forces acting on the control volume, which consist of the body forces acting on the volume and the tractions acting on the surface. For now, we will denote the body forces as simply $\underline{\boldsymbol{f_b}}$ which are essentially the same forces acting on a body in Newton's law, $\underline{\boldsymbol{f}} = m\underline{\boldsymbol{a}}$. The forces acting on the body are conservative, such as gravity which is an example of a conservative force because no dissipation occurs while moving a point mass around a closed loop. Again, we will bring the $\frac{\partial}{\partial t}$ inside of the first RHS term and apply Green's theorem to convert the surface integral into a volume integral.

The surface tractions correspond to surface stresses, that we denote with the stress tensor $\underline{\underline{\boldsymbol{\tau}}}$ which is the total stress. So far we have

$$\int_\Omega \rho\underline{\boldsymbol{f_b}}d\Omega + \int_S \hat{n}\cdot\underline{\underline{\boldsymbol{\tau}}}dS = \int_\Omega \frac{\partial}{\partial t}(\rho\underline{\boldsymbol{u}})d\Omega + \int_\Omega \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})d\Omega \tag{9}$$

We use divergence theorem again on the last remaining surface integral

$$\int_\Omega \rho\underline{\boldsymbol{f_b}}d\Omega + \int_\Omega \underline{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\tau}}}d\Omega = \int_\Omega \frac{\partial}{\partial t}(\rho\underline{\boldsymbol{u}})d\Omega + \int_\Omega \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})d\Omega \tag{10}$$

and can now combine into a single integral for each side

$$\int_\Omega \left(\rho\underline{\boldsymbol{f_b}} + \underline{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\tau}}}\right)d\Omega = \int_\Omega \left(\frac{\partial}{\partial t}(\rho\underline{\boldsymbol{u}}) + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})\right)d\Omega \tag{11}$$

Again, we assert that this must be true for any arbitrary control volume, so we can drop the integral which gives us

$$\rho\underline{\boldsymbol{f_b}} + \underline{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\tau}}} = \frac{\partial}{\partial t}(\rho\underline{\boldsymbol{u}}) + \underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}}) \tag{12}$$

Let's pause for a moment and consider the physical meaning of these terms. The LHS is same as before, these are the body forces acting on the fluid (per unit volume) and the stresses on the surface of the volume (per unit volume). The first term on the RHS is the rate of increase of momentum (per unit volume). The second term is rate of momentum loss by advection through the surface of the volume (per unit volume) and looks like the divergence of kinetic energy. This last expression $(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})$ actually has the form of a tensor, which makes the operation of $\underline{\boldsymbol{\nabla}}\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})$ somewhat difficult to interpret in vector format. However, in indicial notation, the operation is mathematically unambiguous, $\frac{\partial(\rho\underline{\boldsymbol{u_i}}\underline{\boldsymbol{u_j}})}{\partial x_j}$. The expression is called a dyadic product and has the tensor identity:

$$\nabla\cdot(\underline{\boldsymbol{a}}\,\underline{\boldsymbol{b}}) = \underline{\boldsymbol{a}}\,\nabla\underline{\boldsymbol{b}} + \underline{\boldsymbol{b}}\,\nabla\cdot\underline{\boldsymbol{a}} \tag{13}$$

applying this identity to the $(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}})$ term gives

$$\nabla\cdot(\rho\underline{\boldsymbol{u}}\,\underline{\boldsymbol{u}}) = \rho\underline{\boldsymbol{u}}\,\nabla\underline{\boldsymbol{u}} + \underline{\boldsymbol{u}}\,\nabla\cdot(\rho\underline{\boldsymbol{u}}) \tag{14}$$

Substituting these terms into the earlier equation we now have

$$\rho\underline{\boldsymbol{f_b}} + \nabla\cdot\underline{\underline{\boldsymbol{\tau}}} = \frac{\partial(\rho\underline{\boldsymbol{u}})}{\partial t} + \rho\underline{\boldsymbol{u}}\,\nabla\underline{\boldsymbol{u}} + \underline{\boldsymbol{u}}\,\nabla\cdot(\rho\underline{\boldsymbol{u}}) \tag{15}$$

We also know that the conservation of mass holds, and we use its earlier form of

$$\nabla \cdot (\rho \underline{u}) = -\frac{\partial \rho}{\partial t} \tag{16}$$

to substitute into the last term on the RHS from the previous equation, which gives

$$\rho \underline{f_b} + \nabla \cdot \underline{\underline{\tau}} = \frac{\partial(\rho \underline{u})}{\partial t} + \rho \underline{u} \nabla \underline{u} - \underline{u} \frac{\partial \rho}{\partial t} \tag{17}$$

We can expand out the first term on the RHS using the product rule

$$\frac{\partial(\rho \underline{u})}{\partial t} = \rho \frac{\partial \underline{u}}{\partial t} + \underline{u} \frac{\partial \rho}{\partial t} \tag{18}$$

now substituting this back into the previous equation

$$\rho \underline{f_b} + \nabla \cdot \underline{\underline{\tau}} = \rho \frac{\partial \underline{u}}{\partial t} + \underline{u} \frac{\partial \rho}{\partial t} + \rho \underline{u} \nabla \underline{u} - \underline{u} \frac{\partial \rho}{\partial t} \tag{19}$$

now the second and fourth terms cancel while the first and third terms combine to become the material derivative of $\underline{u}$

$$\rho \underline{f_b} + \nabla \cdot \underline{\underline{\tau}} = \rho \frac{D\underline{u}}{Dt} \tag{20}$$

Rearrange the order of the terms and consider their physical meaning.

$$\frac{D(\rho \underline{u})}{Dt} = \nabla \cdot \underline{\underline{\tau}} + \rho \underline{f_b} \tag{21}$$

The LHS is the inertial term and represents the transport of momentum in a fluid, which is described by the material derivative of $\rho \underline{u}$. This is balanced by two physical quantities: 1) the divergence of a source term, $\underline{\underline{\tau}}$, which represents the total stress acting on the body and 2) the body forces to which the fluid is subjected. This is the continuum form of $\underline{f} = m\underline{a}$, for fluids and other continua. [Note, at this stage for a rotating frame, substitute $\underline{u}$ for the appropriate version that includes the angular and translational velocities.]

Constitutive Relationship

The first thing to do is rewrite $\underline{\underline{\tau}}$ (total stress) as two parts, the isotropic (normal) stress and the deviatoric stress arising from shear stresses:

$$\underline{\underline{\tau}} = -P\underline{\underline{I}} + \underline{\underline{\sigma}} \tag{22}$$

where $\underline{\underline{I}}$ is the identity matrix and P is the first invariant of the stress.

In order to describe how the applied deviatoric stress generates deformation of the fluid, we need to know the constitutive relationship between stress and strain rate (i.e. the rheology of the fluid). The theory of Newtonian fluids leads to the following relationship

$$\underline{\underline{\tau}} = \kappa \left(\nabla \cdot \underline{u} \right) \underline{\underline{I}} + \eta \left(\underline{\underline{\nabla u}} + (\underline{\underline{\nabla u}})^T \right) \tag{23}$$

where T indicates the transpose of the tensor, κ is the bulk (or expansion) viscosity and η is the dynamic viscosity. For an incompressible fluids, the divergence of velocity is zero so the

first term vanishes. To gain some insight into the second term, we will use the definition of the deformation tensor, $\underline{\underline{D}}$, or strain rate tensor, which is

$$\underline{\underline{D}} = \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right) \tag{24}$$

it is a symmetric tensor that physically describes the rate of stretching of a fluid. A lot of people write this in indicial notation and it is more commonly called $\underline{\underline{e}}$:

$$\underline{\underline{e}} = \frac{1}{2}\left(\frac{\delta \boldsymbol{u_i}}{\delta x_j} + \frac{\delta \boldsymbol{u_j}}{\delta x_i}\right) \tag{25}$$

In either form, it is also complemented by the antisymmetric friend, the vorticity tensor (or spin tensor)

$$\underline{\underline{W}} = -\frac{1}{2}\left(\boldsymbol{\nabla u} - (\boldsymbol{\nabla u})^T\right) = \frac{1}{2}\left(\frac{\delta \boldsymbol{u_i}}{\delta x_j} - \frac{\delta \boldsymbol{u_j}}{\delta x_i}\right) \tag{26}$$

The vorticity tensor describes the rotational component of deformation. In general, any deformation can be decomposed into two parts, the stretching and the vorticity. Of course, these two tensors can be assembled into a single tensor by simple addition to give $\underline{\underline{L}}$, the velocity gradient tensor

$$\underline{\underline{L}} = \boldsymbol{\nabla u} = \underline{\underline{D}} - \underline{\underline{W}} \tag{27}$$

Substituting back in, everything simplifies down to the usual constitutive relationship

$$\underline{\underline{\sigma}} = 2\eta\underline{\underline{D}} \tag{28}$$

and now we can replace the total stress with the two parts (isotropic and deviatoric)

$$\underline{\underline{\tau}} = -P\underline{\underline{I}} + 2\eta\underline{\underline{D}} \tag{29}$$

Navier Stokes equation

Getting back to the conservation of momentum equation, we can substitute the new expression for total stress into the stress divergence

$$\rho\frac{D\boldsymbol{u}}{Dt} = \nabla \cdot \left(-P\underline{\underline{I}} + \eta\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right)\right) + \rho\boldsymbol{f_b} \tag{30}$$

We need apply the product rule again

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla P + \eta\left(\nabla^2\boldsymbol{u} + \nabla(\nabla \cdot \boldsymbol{u})\right) + \rho\boldsymbol{f_b} \tag{31}$$

once again, notice there is a divergence of velocity, which must be set to zero for incompressible flow and we finally arrive at the Navier-Stokes equation

$$\rho\frac{D\underline{\underline{u}}}{Dt} = -\nabla P + \eta\nabla^2\boldsymbol{u} + \rho\boldsymbol{f_b} \tag{32}$$

dividing through by ρ reveals that the last term on the RHS is actually the diffusion of momentum ($\nu = \eta/\rho$ is the kinematic viscosity which has units of diffusivity, m²s⁻¹)

$$\frac{D\underline{\underline{u}}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f_b} \tag{33}$$

Back up a few steps to when the product rule was applied and notice that it was assumed that viscosity, η, is a constant. This is true for many disciplines that use fluid dynamics, but the one thing that is true for Earth materials is that η is rarely constant because it depends strongly on temperature, pressure, stress, grain size, water content, etc. To be technically correct when implementing this divergence operator, one must be very careful in their treatment of η.

The most common form of the Navier-Stokes equation includes gravity as the only body force

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla P + \nu\nabla^2 u + \underline{g} \tag{34}$$

Scaling analysis and Dynamic similarity

We can gain further insight into the equation by performing a scaling analysis. This allows the equation to become reduced to the least number of controlling parameters. Right now, the key variable, $\underline{u}$, depends on several parameters such as geometry, viscosity, density. If one conducted a systematic study to determine the velocity and varied each of these parameters over some range, it would involve hundreds of experiments. There is some cost associated with each experiment such as your time, the materials, and the chance for experimental error also increases with the number of experiments performed which will hamper the analysis.

The technique for extracting the key controlling parameters of a governing equation is called non-dimensionalization. Each of the variables becomes non-dimensionalized by a characteristic variable. This is a great source of inconsistency amongst all the books, as usually either an asterisk an apostrophe is used and the non-dimensional variables are referred to as the "primed" variables yet other times the variables are renamed as their non-dimensional equivalents and then the dimensional variables are denoted with an asterisk or apostrophe (the latter case is the most confusing..). Here is the definition of all the relevant dimensionless variables *(which I will denote with an asterisk)*

$$u^* = \frac{u}{U}$$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}$$

$$t^* = \frac{tU}{L}, \quad \nabla^* = L\nabla \tag{35}$$

$$P^* = \frac{P-P_0}{\rho U^2} \quad \text{or even better} \quad P^* = \frac{P+\rho g z}{\rho U^2}$$

where P_0 is some representative value of the (modified) pressure in the fluid. There are usually several options for how to non-dimensionalize the pressure term, and the latter choice (using the hydrostatic pressure, $\rho g z$) is an idea of Bernoulli's that is based upon hindsight.

Now we can rewrite the Navier-Stokes equation in terms of the non-dimensionalized quantities and include the force term $\rho\underline{g}$ as $-\rho g\hat{z}$

$$\rho\frac{D u^* U}{Dt^* \frac{L}{U}} = -\frac{1}{L}\nabla^*(\rho U^2 P^* - \rho g z^* L) + \eta\frac{1}{L^2}\nabla^{*2}u^* U - \rho g\hat{z} \tag{36}$$

grouping all the characteristic scale quantities together

$$\frac{\rho U^2}{L}\frac{D u^*}{Dt^*} = -\frac{\rho U^2}{L}\nabla^* P^* + \rho g z^* + \frac{\eta U}{L^2}\nabla^{*2}u^* - \rho g\hat{z} \tag{37}$$

5

Now the usefulness of Bernoulli's hindsight trick is obvious as the body force terms cancel because the unit vector, $\hat{z}$, is also dimensionless. Since the reference pressure, P_0, is an arbitrary choice, one can pick whatever is most convenient. In this case, Bernoulli chose a reference pressure such that it canceled the effect of gravity. Tritton's book has additional insight: "Since the density is assumed to be uniform, the gravitational force is balanced by a vertical pressure gradient which is present whether or not the fluid is moving... This hydrostatic balance can be subtracted out of the dynamical equation.." What becomes obvious is that *only pressure gradients arising from dynamics* are important for the flow. Therefore, P^* can be thought of as a dynamic pressure and the differences in this dynamic pressure result in flow.

Dividing the entire equation through by the quantity, $\rho U^2/L$, gives

$$\frac{D\boldsymbol{u^*}}{Dt^*} = -\nabla^* P^* + \frac{\eta}{\rho U L}\nabla^{*2}\boldsymbol{u^*} \tag{38}$$

This is the non-dimensional form of the Navier-Stokes equation and the non-dimensional group, $\rho U L/\eta$ is known as the Reynolds number, Re

$$\frac{D\boldsymbol{u^*}}{Dt^*} = -\nabla^* P^* + \frac{1}{Re}\nabla^{*2}\boldsymbol{u^*} \tag{39}$$

Before we started, the main variable, $\underline{\boldsymbol{u}}$, depended on at least 3 parameters. But after non-dimensionalization, it is revealed that there is only a single parameter, Re, controlling the governing equation. An important note here is that in order to solve for the flow using the non-dimensional form, the boundary conditions must also be non-dimensionalized using the same process.

The idea of dynamic similarity is that any flow will be similar to any other flow if they have the same non-dimensional number (Re in this case) as well as the same boundary conditions. This is how one can properly scale experiments such that the results of the scaled model are similar to the real case. As an example, in my research I have used fluid dynamic scalings for a penny sinking through a jar of honey to elucidate the fluid dynamics of subducted slab sinking through the Earth's mantle.

Physically, the Reynolds number represents the balance between inertial and viscous forces and this balance completely determines the fluid dynamic regime, whether the flow is viscous ($Re < 1$), laminar ($Re < 2000$) or turbulent ($Re >$ between 2000 and 10^5). For geologic materials with very high viscosities, the viscous forces dominant to the point that the inertial forces can be neglected and this regime ($Re < 1$) is known as Stokes Flow.

There are many other non-dimensional numbers, depending on which forces are included as well as coupling to other governing equations which can produce an equivalent controlling parameter. You may have heard of some of these before, such as the Mach number (Ma), the Ekman number (Ek), the Rossby number (Ro), the Rayleigh number (Ra). Many of the dimensionless groups don't usually play a role in mantle dynamics, like the Froude number, the Weber number, or the Strouhal number, but these play an important role in other fluid dynamic processes.

For a 3-D geometry, we will have 4 unknowns (3 components of velocity and the pressure) and so we need to solve 4 equations. In Cartesian, and going back to the dimensional form, the

equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = -\frac{\partial P}{\partial x} + \eta \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + F_x$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = -\frac{\partial P}{\partial y} + \eta \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] + F_y \tag{40}$$

$$\rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = -\frac{\partial P}{\partial z} + \eta \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] + F_z$$

In geometries other than Cartesian such as Cylindrical polar (r, θ, z) and Spherical polar (r, θ, ϕ), one has to include all the appropriate geometrical terms arising from the gradient and Laplacian operators.

Creeping viscous flow past a sphere

Because the viscous terms dominate, the inertial terms can be neglected and the governing equation is referred to as the Stokes equation

$$0 = -\nabla P^* + \frac{\eta}{\rho U L} \nabla^{*2} \boldsymbol{u^*} \tag{41}$$

in which pressure variations inside the fluid balance the viscous forces.

The most famous application of Stokes flow is that of creeping flow around a sphere. In a laboratory reference frame, the sphere sinks through a viscous fluid and this is actually the fluid dynamics inside a viscometer which is an instrument used to measure viscosity. The solution to the problem of a sinking Stokes sphere is done numerous places (basically every book on fluid dynamics that exists). The version of the solution given below is largely taken from Turcotte and Schubert (1982).

We begin with the dimensional form of the Stokes equation in spherical polar geometry, with the coordinate system that has $\theta = 180°$ in the flow direction, i.e. the fluid approaches the sphere from $z = \infty$ with velocity $-U_0$ in the z-direction. The problem is solved in the reference frame of the sphere (so flow is moving past the sphere) and the sphere has radius a. The problem has an azimuthal symmetry such that $u_\phi = 0$ and $\partial/\partial\phi = 0$.

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta\, u_\theta \right) = 0$$

$$-\frac{\partial P}{\partial r} + \eta \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u_r}{\partial \theta} \right) - \frac{2}{r^2} u_r - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(u_\theta\, \sin\theta \right) \right] = 0$$

$$-\frac{1}{r} \frac{\partial P}{\partial \theta} + \eta \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u_\theta}{\partial \theta} \right) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{1}{r^2 \sin^2\theta} u_\theta \right] = 0 \tag{42}$$

We can solve these equations subject to these 4 boundary conditions

the "no-slip" velocity boundary conditions:
(1) $u_r = 0$ at $r = a$
(2) $u_\theta = 0$ at $r = a$

$$(43)$$

and the "far field" velocity boundary conditions:
(3) $u_r \to -U_0 \cos \theta$ as $r \to \infty$
(4) $u_\theta \to U_0 \sin \theta$ as $r \to \infty$

This is one of those systems of PDE's that is obvious how to solve it when someone else has already found the solution. In any case, let it be said that some pretty smart people have worked on this. According to Turcotte and Schubert, the nature of the boundary conditions suggests that the solution is of the form

$$u_r = f(r) \cos \theta \quad \text{and} \quad u_\theta = g(r) \sin \theta \tag{44}$$

Substituting these functions into the governing equations we obtain

$$\frac{-1}{2r} \frac{d}{dr} \left(r^2 f \right) = g$$

$$-\frac{\partial P}{\partial r} + \frac{\eta \cos \theta}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - 4(f + g) \right] = 0 \tag{45}$$

$$-\frac{\partial P}{\partial \theta} + \frac{\eta \sin \theta}{r} \left[\frac{d}{dr} \left(r^2 \frac{dg}{dr} \right) - 2(f + g) \right] = 0$$

Now apply the $\partial/\partial \theta$ and $\partial/\partial r$ derivatives to the momentum equations for u_r and u_θ, respectively

$$\frac{-1}{2r} \frac{d}{dr} \left(r^2 f \right) = g$$

$$\frac{\partial}{\partial \theta} \left\{ -\frac{\partial P}{\partial r} + \frac{\eta \cos \theta}{r^2} \left[\frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - 4(f + g) \right] \right\} = 0 \tag{46}$$

$$\frac{\partial}{\partial r} \left\{ -\frac{\partial P}{\partial \theta} + \frac{\eta \sin \theta}{r} \left[\frac{d}{dr} \left(r^2 \frac{dg}{dr} \right) - 2(f + g) \right] \right\} = 0$$

and the two momentum equations will be subtracted to eliminate the $\frac{\partial^2 P}{\partial r \partial \theta}$ term which gives us

$$\frac{-1}{2r} \frac{d}{dr} \left(r^2 f \right) = g$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{df}{dr} \right) - \frac{4(f+g)}{r^2} + \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r^2 \frac{dg}{dr} \right) - \frac{2(f+g)}{r} \right] = 0 \tag{47}$$

The solutions of functions f and g can be found by assuming simple powers of r

$$f = c r^n \tag{48}$$

where c is a constant. Substituting this into the continuity equation gives

$$g = \frac{-c(n + 2)}{2} r^n \tag{49}$$

8

Now the functions of f and g can be substituted into the remaining momentum equation and it produces a simple algebraic expression which has several roots for n

$$n(n+3)(n-2)(n+1) = 0 \text{ which gives } n = 0, -3, 2, -1 \tag{50}$$

this gives the full description for the linear combinations of $f(r)$ and $g(r)$ using the values of n that were determined

$$f = c_1 + \frac{c_2}{r^3} + \frac{c_3}{r} + c_4 r^2$$

$$g = -c_1 + \frac{c_2}{2r^3} - \frac{c_3}{2r} - 2c_4 r^2 \tag{51}$$

These can be substituted into the expressions for velocity to give

$$u_r = \left(c_1 + \frac{c_2}{r^3} + \frac{c_3}{r} + c_4 r^2\right)\cos\theta$$

$$u_\theta = \left(-c_1 + \frac{c_2}{2r^3} - \frac{c_3}{2r} - 2c_4 r^2\right)\sin\theta \tag{52}$$

We can start to apply the boundary conditions to solve for the constants. Applying the far field velocity boundary conditions gives

$$c_1 = -U_0 \text{ and } c_4 = 0 \tag{53}$$

Applying the no-slip condition at $r = a$ gives

$$c_2 = \frac{-a^3 U_0}{2} \text{ and } c_3 = \frac{3a U_0}{2} \tag{54}$$

This gives the final expressions for the velocity components

$$u_r = -U_0\left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right)\cos\theta$$

$$u_\theta = U_0\left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)\sin\theta \tag{55}$$

These can be substituted back into the original equation for momentum in the θ direction and integrating with respect to θ

$$P = \frac{3\eta a U_0}{2r^2}\cos\theta \tag{56}$$

The solution for flow is now given as both components of velocity as well as pressure have been solved. We can learn even more about these fluid dynamics if we consider the physics near the surface of the sphere. Stokes flow is a balance of viscous forces and pressure and the net effect of these forces describes the amount of drag the sphere has with respect to the surrounding flow. Since we know the solution to the flow, we will can calculate these forces and determine the drag on the sphere. There are two contributions to the drag, one from pressure and one from viscous stresses.

$$D = D_P + D_v \tag{57}$$

In order to calculate the contribution from pressure, we need the component of the force in the direction that pressure is pushing on the sphere. This equates to the vertical component of the pressure in the negative z direction projected on the surface of the sphere at radius a, or just

$$P\cos\theta = \frac{3\eta U_0}{2a}\cos^2\theta \tag{58}$$

9

We need to integrate this pressure over the surface of the sphere, but since it only acts on the cross-sectional area of the sphere ($\pi a^2 \sin \theta$) we have

$$D_P = \int_0^\pi (P \cos \theta) \, 2\pi a^2 \sin \theta d\theta = 3\pi \eta a U_0 \int_0^\pi \sin \theta \cos^2 \theta d\theta = 2\pi \eta a U_0 \tag{59}$$

The viscous contribution to the drag has two components, one from the normal stresses and one from the tangential stress, so we need to apply the constitutive relation ($\sigma = \eta \dot{\varepsilon}$) using the strain rates in spherical polar coordinates

$$(\sigma_{rr})_{r=a} = 2\eta \left(\frac{\partial u_r}{\partial r} \right)_{r=a}$$

$$(\sigma_{r\theta})_{r=a} = \eta \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right)_{r=a} \tag{60}$$

These are easily found by substituting in the solutions for the velocity components

$$(\sigma_{rr})_{r=a} = 0$$

$$(\sigma_{r\theta})_{r=a} = \frac{3\eta U_0 \sin \theta}{2a} \tag{61}$$

There are no normal stresses because the sphere is defined to rigid. It is a property of incompressible fluid that the deviatoric stress acting across a rigid boundary is wholly tangential. The tangential stress is in the θ direction all along the sphere, but we need the component in the negative z direction so use the $\sin \theta$ projection

$$\sigma_{r\theta} \sin \theta = \frac{3\eta U_0 \sin^2 \theta}{2a} \tag{62}$$

Once again, integrate the product of this quantity with the cross-sectional area of the sphere

$$D_v = \int_0^\pi \left(\frac{3\eta U_0 \sin^2 \theta}{2a} \right) 2\pi a^2 \sin \theta d\theta = 3\pi \eta a U_0 \int_0^\pi \sin^3 \theta d\theta = 4\pi \eta a U_0 \tag{63}$$

Notice that the contribution to drag from viscous stresses is exactly double the contribution from pressure forces. It is more common to report the drag coefficient, c_D, defined the total drag normalized by both a characteristic pressure ($\frac{1}{2}\rho_f U_0^2$) and cross-sectional area of the sphere (πa^2)

$$c_D \equiv \frac{D}{\frac{1}{2}\rho U_0^2 \pi a^2} = \frac{D_P + D_v}{\frac{1}{2}\rho U_0^2 \pi a^2} = \frac{6\pi \eta a U_0}{\frac{1}{2}\rho U_0^2 \pi a^2} = \frac{12}{(\rho U_0 a)/\eta} = \frac{24}{Re} \tag{64}$$

Notice that the Reynolds number appears in the denominator. These sinking sphere experiments can be done at various Re and it is very striking that the predicted Stokes drag coefficient holds remarkably well up until $Re \sim 1$ when inertial effects begin to become important.

The final thing that is useful to do is calculate the terminal velocity of the sphere. As the sphere can be rising or sinking, it has many applications in geological fluid dynamics such as settling of crystals in a magma, or rise of a plume head in the mantle. Archimedes principle describes the buoyancy force of an object as the density contrast with respect to a background fluid, in this case a rising sphere

$$F = (\rho_f - \rho_s)g \left(\frac{4}{3}\pi a^2 \right) \tag{65}$$

and by setting this force equal to the drag force, the terminal upward velocity is obtained.

$$U = \frac{a^2 (\rho_f - \rho_s)g}{9\eta} \tag{66}$$

It is important to recognize that the velocity depends on the radius squared.

10